

WEIGHTED NORM INEQUALITIES, OFF-DIAGONAL ESTIMATES AND ELLIPTIC OPERATORS

PART I: GENERAL OPERATOR THEORY AND WEIGHTS

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ABSTRACT. This is the first part of a series of four articles. In this work, we are interested in weighted norm estimates. We put the emphasis on two results of different nature: one is based on a good- λ inequality with two-parameters and the other uses Calderón-Zygmund decomposition. These results apply well to singular “non-integral” operators and their commutators with bounded mean oscillation functions. Singular means that they are of order 0, “non-integral” that they do not have an integral representation by a kernel with size estimates, even rough, so that they may not be bounded on all L^p spaces for $1 < p < \infty$. Pointwise estimates are then replaced by appropriate localized $L^p - L^q$ estimates. We obtain weighted L^p estimates for a range of p that is different from $(1, \infty)$ and isolate the right class of weights. In particular, we prove an extrapolation theorem “à la Rubio de Francia” for such a class and thus vector-valued estimates.

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GENERAL INTRODUCTION

This is a general introduction for this article and the series [AM2, AM3, AM4].

Calderón-Zygmund operators have been thoroughly studied since the 50's. They are singular integral operators associated with a kernel satisfying certain size and smoothness conditions. One first shows that the operator in question is bounded on L^{p_0} for some p_0 : typically, for $p_0 = 2$ with spectral theory, Fourier transform or even the powerful $T(1)$, $T(b)$ theorems. Once this is achieved, using the properties of the kernel, one gets a weak-type (1,1) estimate hence strong type (p, p) for $1 < p < p_0$ by means of the Calderón-Zygmund decomposition and for $p > p_0$, one uses duality or boundedness from L^∞ to BMO and interpolation. Still another way for $p > p_0$ relies on good- λ estimates via the Fefferman-Stein sharp maximal function. It is interesting to note that both Calderón-Zygmund decomposition and good- λ arguments use independent smoothness conditions on the kernel, allowing generalizations in various ways.

The removal of regularity assumptions on the kernel is important, for instance towards applications to operators on non-smooth domains. Let us mention [DMc] where a weak-type (1,1) criterion is obtained under upper bound assumption on the kernel but no regularity in the classical Hölder sense or in the sense of the Hörmander condition [Hör].

We mention also that Calderón-Zygmund operators satisfy also commutator estimates with bounded mean oscillation functions and it is therefore natural to try to extend them (see also the work of [DY] in this direction following the methods in [DMc]).

A natural question is in what sense one should use the kernel of the operators. It has become common practice but is it a necessary limitation or a technical one. Indeed, one encounters Calderón-Zygmund like operators without any (reasonable) information on their kernels which we call, following the implicit terminology introduced in [BK1], singular “non-integral” operators in the sense that they are still of order 0 but they do not have an integral representation by a kernel with size and/or smoothness estimates. The goal is to obtain some range of exponents p for which L^p boundedness holds, and because this range may not be $(1, \infty)$, one should abandon any use of kernels.

The first step was done in [BK1] where a criterion for weak-type (p, p) for some $p < p_0$ is presented. In fact, this criterion is in the air in [Fef] but, still, [BK1] brings some novelty such as the removal of the mean value property already observed in [DMc] when $p = 1$. See also [BK2] and [HM] for L^p bounds $p < 2$ of the Riesz transforms of elliptic operators starting from the L^2 bound proved in [AHLMcT].

The second step was taken in [ACDH], inspired by the good- λ estimates in the Ph.D. thesis of one of us [Ma1, Ma2], where a criterion for strong type (p, p) for *some* $p > p_0$ is proved and applied to Riesz transforms for the Laplace-Beltrami operators on some Riemannian manifolds. A criterion in the same spirit for a limited range of p 's also appears implicitly in [CP] towards perturbation theory for linear and non-linear elliptic equations and more explicitly in [Sh1, Sh2] (actually, we shall observe here that the criterion in [Sh2] is a corollary of the one in [ACDH]).

These two criteria are exposed in [Au1], to which the reader is referred, in the Euclidean setting and applied to other operators.

Our purpose is to investigate the weighted norm counterparts of this new theory for Muckenhoupt weights and to apply this in the subsequent papers. Again, the weighted norm theory is well known for Calderón-Zygmund operators and we seek for criteria applying to larger classes of operators *without kernel bounds* hence with limited range of exponents. We mention [Ma1] where some weighted estimates for a functional calculi are proved but again assuming appropriate kernel upper bounds. Our study will also clarify some points in the unweighted case: in particular, we present a simple machinery to prove (new) commutator estimates (both unweighted and weighted) in this generality.

This paper is concerned with the general operator theory and weights in the setting of spaces of homogeneous type. We study weighted boundedness criteria for operators and their commutators with bounded mean oscillation functions. Available machinery give us also vector-valued estimates. See the specific introductions of Parts 1 and 2 in this paper.

Part II, [AM2], is of independent interest as it develops a theory of off-diagonal estimates in the context of spaces of homogeneous type. In particular, the case of the semigroups generated by elliptic operators is thoroughly studied. This is instrumental in the application of the general theory in [AM3].

In Part III, [AM3], we consider operators arising from second order elliptic operators L : operators of the type $\varphi(L)$ from holomorphic functional calculus, the Riesz transforms, square functions, We obtain sharp or nearly sharp ranges of weighted boundedness of such operators, of their commutators with bounded mean oscillation functions, and also vector-valued inequalities.

In Part IV, [AM4], we apply our general theory to the Riesz transform on some Riemannian manifolds or Lie groups as in [ACDH] and their commutators.

Part 1. Good- λ methods

1. INTRODUCTION

Good- λ inequalities, brought to Harmonic Analysis in [BG], provide a powerful tool to prove boundedness results for operators or at least comparisons of two operators. A typical good- λ inequality for two non-negative functions F and G is as follows: for every $0 < \delta < 1$ there exists $\gamma = \gamma(\delta)$ and for every $w \in A_\infty$, there exists $0 < \epsilon_w \leq 1$

and $C_w > 0$ such that for any $\lambda > 0$

$$w\{x : F(x) > 2\lambda, G(x) \leq \gamma\lambda\} \leq C_w \delta^{\epsilon_w} w\{x : F(x) > \lambda\}. \quad (1.1)$$

The usual approach for proving such an estimate consists in first deriving a local version of it with respect to the underlying doubling measure, and then passing to the weighted measure using that $w \in A_\infty$.

Weighted good- λ estimates encode a lot of information about F and G , since they give a comparison of the w -measure of the level sets of both functions. As a consequence of (1.1) one gets, for instance, that for every $0 < p < \infty$ and all $w \in A_\infty$ then $\|F\|_{L^p(w)}$ is controlled by $\|G\|_{L^p(w)}$. The same inequality holds with $L^{p,\infty}$ in place of L^p or with some other function spaces. Thus, the size of F is controlled by that of G .

In applications, one tries to control a specific operator T to be studied by a maximal one M whose properties are known by setting $F = Tf$ and $G = Mf$. For example, a Calderón-Zygmund operator by the Hardy-Littlewood maximal operator [Coi], [CF]; a fractional integral by a fractional maximal operator [MW]; a Littlewood-Paley square function by a non-tangential maximal operator [CWW], [Dah], [DJK], [GW], [Wil]; the maximal operator by the sharp maximal operator [FS].

When T is a Calderón-Zygmund operator with smooth kernel, in particular it is already bounded on (unweighted) L^2 , it was shown in [Coi], [CF] that (1.1) holds with $F = Tf$ and $G = Mf$ with M being the Hardy-Littlewood maximal function. Thus, T is “controlled” by M in $L^p(w)$ for all $0 < p < \infty$ and $w \in A_\infty$ and therefore T is bounded on $L^p(w)$ if M is bounded on $L^p(w)$, which by Muckenhoupt’s theorem means $w \in A_p$. In particular, the range of unweighted L^p boundedness of T , that is the set of p for which T is strong-type (p, p) , is $(1, \infty)$, a fact that was known by Calderón-Zygmund methods (see Part 2 of this paper).

Replacing Mf by $M(|f|^{p_0})^{1/p_0}$ for some $p_0 > 1$ changes the range of unweighted L^p boundedness to (p_0, ∞) . See for instance [MPT], and the references therein, where this occurs for Calderón-Zygmund operators with less regular kernels. In this case, weighted $L^p(w)$ boundedness holds if $w \in A_{p/p_0}$.

So far, there is a lower limitation on p but no upper limitation in the sense that p goes all the way to ∞ . This has to be so by a special and very simple case of Rubio de Francia’s extrapolation theorem (see [Rub], [Gar]) which says that any sublinear operator T that is bounded on $L^{p_1}(w)$ for some $0 < p_1 < \infty$ and all $w \in A_1$, is bounded on L^p for all $p_1 \leq p < \infty$.

Obviously, the above good- λ inequality does not apply to operators whose L^p boundedness is expected for $p_0 < p < q_0$ with a finite exponent q_0 . An example is the Riesz transform for the Laplace-Beltrami operator on some Riemannian manifolds studied in [ACDH, AC]. There, a two-parameter good- λ estimate incorporating an upper limitation in p is used for proving L^p boundedness with a limited range of $p > 2$. See also [Au1], [CP], [Sh1, Sh2]. These two-parameter good- λ estimates are of the form

$$|\{x : MF(x) > K\lambda, G(x) \leq \gamma\lambda\}| \leq C \left(\frac{1}{K^{q_0}} + \frac{\gamma}{K} \right) |\{x : MF(x) > \lambda\}|, \quad (1.2)$$

for all $\lambda > 0$, $K \geq K_0$ and $0 < \gamma < 1$. Note the explicit dependance on K, γ which are the two parameters and the appearance of the exponent $q_0 \in (0, \infty]$ in the right hand side. From this, it follows that MF is controlled by G in L^p for all $0 < p < q_0$.

The aim of this part is to state conditions to obtain a weighted analog of (1.2) and to derive some consequences for the study of operators. As we see below (Section 3.1), this forces us to specify the power ϵ_w in (1.1), hence to specify the reverse Hölder class for w . Indeed, taking $w \in RH_{s'}$ then $\epsilon_w = 1/s$ and we obtain the control of MF by G in $L^p(w)$ for all $0 < p < q_0/s$ (note that this implies that $w \in RH_{(q_0/p)'}).$

This allows us to formulate simple unweighted conditions for the $L^p(w)$ boundedness of (singular “non-integral”) operators a priori bounded on (unweighted) L^p for $p_0 < p < q_0$ for weights in the class $\mathcal{W}^p(p_0, q_0) = A_{p/p_0} \cap RH_{(q_0/p)'}$ (Section 3.4). A slight improvement furnishes, almost for free, boundedness of their commutators with bounded mean oscillation functions for the same weights (Section 3.5). This class of weights (studied in Section 4.1) is the largest possible within A_∞ as we prove an extrapolation result for it. Namely, if T is bounded on some $L^p(w)$ for some fixed p and for all $w \in \mathcal{W}^p(p_0, q_0)$, then the same happens for every $q \in (p_0, q_0)$ and the corresponding class of weights. Using ideas on extrapolation from [CMP] and [CGMP], we obtain vector-valued inequalities automatically again for limited ranges of p (Section 4.2). For simplicity of the exposition, we work in the Euclidean space equipped with the Lebesgue measure. See Section 5 for extensions to spaces of homogeneous type.

2. MUCKENHOUT WEIGHTS

We review some needed background on Muckenhoupt weights. We use the notation

$$\oint_E h = \frac{1}{|E|} \int_E h(x) dx$$

and we often forget the Lebesgue measure and the variable of the integrand in writing integrals, unless this is needed to avoid confusions.

A weight w is a non-negative locally integrable function. We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$ (balls could be switched to cubes)

$$\left(\oint_B w \right) \left(\oint_B w^{1-p'} \right)^{p-1} \leq C.$$

For $p = 1$, we say that $w \in A_1$ if there is a constant C such that for every ball $B \subset \mathbb{R}^n$

$$\oint_B w \leq C w(x), \quad \text{for a.e. } x \in B,$$

or, equivalently, $Mw \leq Cw$ a.e. where M denotes the uncentered maximal operator over balls (or cubes) in \mathbb{R}^n . The reverse Hölder classes are defined in the following way: $w \in RH_q$, $1 < q < \infty$, if there is a constant C such that for every ball $B \subset \mathbb{R}^n$

$$\left(\oint_B w^q \right)^{\frac{1}{q}} \leq C \oint_B w.$$

The endpoint $q = \infty$ is given by the condition: $w \in RH_\infty$ whenever, for any ball B ,

$$w(x) \leq C \oint_B w, \quad \text{for a.e. } x \in B.$$

Notice that we have excluded the case $q = 1$ since the class RH_1 consists of all the weights, and that is the way RH_1 is understood in what follows.

We sum up some of the properties of these classes in the following result.

Proposition 2.1.

- (i) $A_1 \subset A_p \subset A_q$ for $1 \leq p \leq q < \infty$.
 - (ii) $RH_\infty \subset RH_q \subset RH_p$ for $1 < p \leq q \leq \infty$.
 - (iii) If $w \in A_p$, $1 < p < \infty$, then there exists $1 < q < p$ such that $w \in A_q$.
 - (iv) If $w \in RH_q$, $1 < q < \infty$, then there exists $q < p < \infty$ such that $w \in RH_p$.
 - (v) $A_\infty = \bigcup_{1 \leq p < \infty} A_p = \bigcup_{1 < q \leq \infty} RH_q$.
 - (vi) If $1 < p < \infty$, $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$.
 - (vii) If $1 \leq q \leq \infty$ and $1 \leq s < \infty$, then $w \in A_q \cap RH_s$ if and only if $w^s \in A_{s(q-1)+1}$.
- Properties (i)-(vi) are standard, see for instance [GR] or [Duo]. For (vii) see [JN].

3. TWO PARAMETER GOOD- λ ESTIMATES

Unless specified otherwise, M denotes the uncentered maximal operator over cubes (or balls) in \mathbb{R}^n .

3.1. Main result.

Theorem 3.1. Fix $1 < q \leq \infty$, $a \geq 1$ and $w \in RH_{s'}$, $1 \leq s < \infty$. Then, there exist $C = C(q, n, a, w, s)$ and $K_0 = K_0(n, a) \geq 1$ with the following property: Assume that F , G , H_1 and H_2 are non-negative measurable functions on \mathbb{R}^n such that for any cube Q there exist non-negative functions G_Q and H_Q with $F(x) \leq G_Q(x) + H_Q(x)$ for a.e. $x \in Q$ and

$$\left(\int_Q H_Q^q \right)^{\frac{1}{q}} \leq a (MF(x) + MH_1(x) + H_2(\bar{x})), \quad \forall x, \bar{x} \in Q; \quad (3.1)$$

and

$$\int_Q G_Q \leq G(x), \quad \forall x \in Q. \quad (3.2)$$

Then for all $\lambda > 0$, $K \geq K_0$ and $0 < \gamma < 1$

$$w\{MF > K\lambda, G + H_2 \leq \gamma\lambda\} \leq C \left(\frac{a^q}{K^q} + \frac{\gamma}{K} \right)^{\frac{1}{s}} w\{MF + MH_1 > \lambda\}. \quad (3.3)$$

As a consequence, for all $0 < p < \frac{q}{s}$, we have

$$\|MF\|_{L^p(w)} \leq C (\|G\|_{L^p(w)} + \|MH_1\|_{L^p(w)} + \|H_2\|_{L^p(w)}), \quad (3.4)$$

provided $\|MF\|_{L^p(w)} < \infty$, and

$$\|MF\|_{L^{p,\infty}(w)} \leq C (\|G\|_{L^{p,\infty}(w)} + \|MH_1\|_{L^{p,\infty}(w)} + \|H_2\|_{L^{p,\infty}(w)}), \quad (3.5)$$

provided $\|MF\|_{L^{p,\infty}(w)} < \infty$. Furthermore, if $p \geq 1$ then (3.4) and (3.5) hold, provided $F \in L^1$ (whether or not $MF \in L^p(w)$).

The proof of this result is in Section 6.1.

Remark 3.2. We do mean that the estimates (3.1) and (3.2) are valid at *any* points $x, \bar{x} \in Q$, not just almost everywhere.

Remark 3.3. The case $q = \infty$ is the standard one: the L^q -average appearing in the hypothesis is understood as an essential supremum and $K^{-q} = 0$. Thus, the $L^p(w)$ and $L^{p,\infty}(w)$ estimates will hold for any $0 < p < \infty$, no matter the value of s , that is, for any $w \in A_\infty$.

Remark 3.4. If (3.1) holds for any $q > 1$, then (3.4) holds for all $0 < p < \infty$ and for all $w \in A_\infty$. To see this, we fix $0 < p < \infty$ and $w \in A_\infty$. Then $w \in RH_{s'}$ for some $1 \leq s < \infty$ and it suffices to take q large enough so that $p < q/s$.

Remark 3.5. In applications, error terms appear in localization arguments either in the form $MH_1(x)$ or $H_2(\bar{x})$ (with \bar{x} independent of x) or both. The unweighted case [ACDH, Theorem 2.4] is of this type.

Remark 3.6. If $s > 1$ and $q < \infty$, then one also obtains the end-point $p = q/s$. To do it, we only need to observe that $w \in RH_{s'_0}$ for some $1 < s_0 < s$ (see (v) in Proposition 2.1) and so we can apply Theorem 3.1 with $p = q/s < q/s_0$.

We present some applications of Theorem 3.1 recovering some previously known estimates.

3.2. Fefferman-Stein Inequality. The classical Fefferman-Stein inequality relating M and $M^\#$ follows at once from Theorem 3.1. We take $F = |f| \in L^1_{\text{loc}}$, $H_1 = H_2 = 0$. For each cube Q we denote by f_Q the average of f on Q ,

$$F = |f| \leq |f_Q| + |f - f_Q| \equiv H_Q + G_Q.$$

Taking $q = \infty$, we trivially have $\|H_Q\|_{L^\infty(Q)} = |f_Q| \leq Mf(x) = MF(x)$ for each $x \in Q$. Also, by definition of $M^\#$

$$\int_Q G_Q = \int_Q |f - f_Q| \leq M^\# f(x) \equiv G(x), \quad \forall x \in Q.$$

Thus, (3.3) holds (with $q = \infty$) and consequently, for every $0 < p < \infty$ and every $w \in A_\infty$ we have

$$\|Mf\|_{L^p(w)} \leq C \|M^\# f\|_{L^p(w)}, \quad (3.6)$$

whenever $Mf \in L^p(w)$. This is what is proved in [FS].

3.3. Generalized sharp maximal functions. In [Ma1], a generalization of $M^\#$ is introduced in the setting of spaces of homogeneous type. In the Euclidean setting, we define $M_D^\#$ as follows. Let $\{D_t\}_{t>0}$ be a family of operators (for instance, an approximation of the identity but it could be more general) such that each D_t is an integral operator with kernel $d_t(x, y)$ for which

$$|d_t(x, y)| \leq C t^{-\frac{n}{m}} h(|x - y|^m t^{-1})$$

where m is some positive fixed constant and h is positive, bounded, decreasing and decaying to 0 fast enough. Then we define a new sharp maximal function associated to $\{D_t\}_{t>0}$ as

$$M_D^\# f(x) = \sup_{Q \ni x} \int_Q |f - D_{t_Q} f|$$

where $t_Q = \ell(Q)^m$ and $\ell(Q)$ is the sidelength of Q .

Examples are given by the semigroups associated with a second order elliptic operators $\{e^{-tL}\}_{t>0}$ whose heat kernels have Gaussian (or some other) decay (see [AT, Ma1, DMc, AE], ...)

With Theorem 3.1 we can reprove the good- λ inequality of [Ma1] for $M_D^\#$ and M . As before take $F = |f| \in L^p$ for some $p \geq 1$, $H_1 = H_2 = 0$. For each cube Q we write

$$F = |f| \leq |D_{t_Q} f| + |f - D_{t_Q} f| \equiv H_Q + G_Q.$$

Taking $q = \infty$, we have $\|H_Q\|_{L^\infty(Q)} \leq C M f(x) = C M F(x)$ for each $x \in Q$ by the properties assumed on D_t . Moreover, by definition of $M_D^\#$,

$$\int_Q G_Q = \int_Q |f - D_{t_Q} f| \leq M_D^\# f(x) \equiv G(x), \quad \forall x \in Q.$$

Thus, one obtains (3.3) (with $q = \infty$) and hence, for every $0 < p < \infty$ and every $w \in A_\infty$ we have

$$\|Mf\|_{L^p(w)} \leq C \|M_D^\# f\|_{L^p(w)},$$

whenever $Mf \in L^p(w)$. This is the result proved in [Ma1].

3.4. Applications to Singular “Non-Integral” Operators. We present here different applications of Theorem 3.1 toward weighted norm inequalities for operators, avoiding all use of kernel representation, hence the terminology “non-integral”.

In what follows, we say that an operator T acts from A into B (with A, B being some given sets) if T is a map defined on A and valued in B . An operator T acting from A to B , both vector spaces of measurable functions, is sublinear if

$$|T(f+g)| \leq |Tf| + |Tg| \quad \text{and} \quad |T(\lambda f)| = |\lambda| |Tf|$$

for all $f, g \in A$ and $\lambda \in \mathbb{R}$ or \mathbb{C} . Let us mention that for the theorems of this section, the second condition is not needed.

Theorem 3.7. *Let $1 \leq p_0 < q_0 \leq \infty$. Let \mathcal{E} and \mathcal{D} be vector spaces such that $\mathcal{D} \subset \mathcal{E}$. Let T, S be operators such that S acts from \mathcal{D} into the set of measurable functions and T is sublinear acting from \mathcal{E} into L^{p_0} . Let $\{\mathcal{A}_r\}_{r>0}$ be a family of operators acting from \mathcal{D} into \mathcal{E} . Assume that*

$$\left(\int_B |T(I - \mathcal{A}_{r(B)})f|^{p_0} \right)^{\frac{1}{p_0}} \leq C M(|Sf|^{p_0})^{\frac{1}{p_0}}(x), \quad (3.7)$$

and

$$\left(\int_B |T\mathcal{A}_{r(B)}f|^{q_0} \right)^{\frac{1}{q_0}} \leq C M(|Tf|^{p_0})^{\frac{1}{p_0}}(x), \quad (3.8)$$

for all $f \in \mathcal{D}$, all ball B where $r(B)$ denotes its radius and all $x \in B$. Let $p_0 < p < q_0$ (or $p = q_0$ when $q_0 < \infty$) and $w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{p})}'$. There is a constant C such that

$$\|Tf\|_{L^p(w)} \leq C \|Sf\|_{L^p(w)} \quad (3.9)$$

for all $f \in \mathcal{D}$. Furthermore, for all $p_0 < r < q_0$, there is a constant C such that

$$\left\| \left(\sum_j |Tf_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)} \leq C \left\| \left(\sum_j |Sf_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)} \quad (3.10)$$

for all $f_j \in \mathcal{D}$.

We would like to emphasize that (3.7) and (3.8) are unweighted assumptions. This is a triple extension of [ACDH, Theorem 3.1]: we introduce a second operator S , obtain weighted inequalities and also vector-valued estimates.

Remark 3.8. The most common situation is $S = I$, $\mathcal{E} = L^{p_0}$ with \mathcal{D} being a class of “nice” functions such as L^{p_0} , $L^{p_0} \cap L^2$, L_c^∞ , C_0^∞ , \dots . In that case, (3.9) is interesting only when the right hand side is finite, hence we may also impose $f \in L^p(w)$. This implies the boundedness of T from $\mathcal{D} \cap L^p(w)$ into $L^p(w)$ for the $L^p(w)$ norm. See [AM3] for a situation where $S \neq I$.

Remark 3.9. In this result, the case $q_0 = \infty$ is understood in the sense that the L^{q_0} -average in (3.8) is indeed an essential supremum. Besides, the condition for the weight turns out to be $w \in A_{p/p_0}$ for $p > p_0$. Similarly, if (3.8) is satisfied for all $q_0 < \infty$ then (3.9) holds for all $p_0 < p < \infty$ and for all $w \in A_{p/p_0}$.

Remark 3.10. A slightly more general statement consists in replacing the family $\{\mathcal{A}_r\}$ by $\{\mathcal{A}_B\}$ indexed by balls. We use this below.

Proof of Theorem 3.7. The vector-valued inequalities (3.10) follow automatically by extrapolation, see Theorem 4.9 below.

We prove (3.9), first in the case $q_0 < \infty$ and $p_0 < p \leq q_0$. Let $f \in \mathcal{D}$ and so $F = |Tf|^{p_0} \in L^1$. Fix a cube Q (we switch to cubes for the proof). As T is sublinear, we have

$$F \leq G_Q + H_Q \equiv 2^{p_0-1} |T(I - \mathcal{A}_{r(Q)})f|^{p_0} + 2^{p_0-1} |T\mathcal{A}_{r(Q)}f|^{p_0}.$$

Then (3.7) and (3.8) yield the corresponding conditions (3.1) and (3.2) with $q = q_0/p_0$, $H_1 = H_2 \equiv 0$, $a = 2^{p_0-1} C^{p_0}$ and $G = 2^{p_0-1} C^{p_0} M(|Sf|^{p_0})$. As $w \in RH_{(q_0/p)'}'$, Theorem 3.1 and Remark 3.6 (since $q_0 < \infty$ implies $q < \infty$) with $p/p_0 > 1$ in place of p and $s = q_0/p$ yield

$$\|Tf\|_{L^p(w)}^{p_0} \leq \|MF\|_{L^{\frac{p}{p_0}}(w)} \leq C \|G\|_{L^{\frac{p}{p_0}}(w)} = C \|M(|Sf|^{p_0})\|_{L^{\frac{p}{p_0}}(w)} \leq C \|Sf\|_{L^p(w)}^{p_0},$$

where in the last estimate we have used that $w \in A_{p/p_0}$.

In the case $q_0 = \infty$ and $p < \infty$, Theorem 3.1 applies as before when $w \in A_{p/p_0}$ by Remark 3.3. \square

Remark 3.11. Under the assumptions of Theorem 3.7, we can also prove an end-point weak-type estimate. Namely, if $w \in A_1 \cap RH_{(\frac{q_0}{p_0})}'$, then there is a constant C such that

$$\|Tf\|_{L^{p_0,\infty}(w)} \leq C \|Sf\|_{L^{p_0}(w)}, \quad (3.11)$$

for all $f \in \mathcal{D}$. The proof follows the same ideas but one has to use the weak-type estimate (3.5) in place of (3.4). The details are left to the reader.

Let us recall that we have assumed that for $f \in \mathcal{D}$ then $F = |Tf|^{p_0} \in L^1$. This hypothesis is not granted directly for T in some applications (for instance, it is not true for $p_0 = 1$ and T being the Hilbert transform or the Riesz transforms) but for suitable approximations T_ε that are bounded on $L^{p_0}(w)$ (with some bound that is allowed to depend on ε). In such a case, one obtains the weak-type estimate for T_ε with a uniform control on the constant and the weak-type estimate for T follows by a limiting procedure. (This happens for the Hilbert transform: the kernel is truncated in such a way that it is in L^1 , so the approximations T_ε are bounded on L^1 .) Let us mention that for Calderón-Zygmund operators the usual approach is different: the weighted weak-type $(1,1)$ estimate for A_1 weights follows by using the Calderón-Zygmund decomposition (see [GR, Chapter IV]), see also [BK1] for a weak-type (p_0, p_0) with $p_0 > 1$, and Theorems 8.1, 8.7 below.

Remark 3.12. Theorem 3.1 implies a variant of Theorem 3.7 valid for all $0 < p_0 < q_0 \leq \infty$. We do not know, however, whether such a result is useful in applications when $p_0 < 1$. The precise statement and the minor modifications in the proof are left to the reader.

The following extension of Theorem 3.7 is also useful. For simplicity we assume that $S = I$.

Theorem 3.13. *Let $1 \leq p_0 < q_0 \leq \infty$. Let \mathcal{D} , \mathcal{E} , T and $\{\mathcal{A}_r\}_{r>0}$ be as in Theorem 3.7. Assume that (3.7) holds with $S = I$ and, in place of (3.8), that*

$$\left(\int_B |T\mathcal{A}_r(B)f|^{q_0} \right)^{\frac{1}{q_0}} \leq C \left(M(|Tf|^{p_0})^{\frac{1}{p_0}}(x) + M(|S_1 f|^{p_0})^{\frac{1}{p_0}}(x) + |S_2 f(\bar{x})| \right), \quad (3.12)$$

holds for all $f \in \mathcal{D}$ and all $x, \bar{x} \in B$ where S_1, S_2 are two given operators. Let $p_0 < p < q_0$ and $w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{p})}'$. If S_1 and S_2 are bounded on $L^p(w)$, then

$$\|Tf\|_{L^p(w)} \leq C \|f\|_{L^p(w)}$$

for all $f \in \mathcal{D} \cap L^p(w)$.

Observe that Remarks 3.9 and 3.10 apply to this result. Also, the operator T satisfies the vector-valued inequalities (3.10).

Proof. The proof is almost identical to the one of Theorem 3.7. Let $f \in \mathcal{D} \cap L^p(w)$ and set $F = |Tf|^{p_0} \in L^1$, $H_1 = |S_1 f|^{p_0}$ and $H_2 = |S_2 f|^{p_0}$. Theorem 3.1 gives us

$$\begin{aligned} \|Tf\|_{L^p(w)}^{p_0} &\leq \|MF\|_{L^{\frac{p}{p_0}}(w)} \leq C \left(\|G\|_{L^{\frac{p}{p_0}}(w)} + \|MH_1\|_{L^{\frac{p}{p_0}}(w)} + \|H_2\|_{L^{\frac{p}{p_0}}(w)} \right) \\ &= C \left(\|M(|Sf|^{p_0})\|_{L^{\frac{p}{p_0}}(w)} + \|M(|S_1 f|^{p_0})\|_{L^{\frac{p}{p_0}}(w)} + \| |S_2 f|^{p_0} \|_{L^{\frac{p}{p_0}}(w)} \right) \\ &\leq C \|f\|_{L^p(w)}^{p_0}, \end{aligned}$$

where we have used that M is bounded on $L^{\frac{p}{p_0}}(w)$ (since $w \in A_{p/p_0}$) and that, by hypothesis, S_1, S_2 are bounded on $L^p(w)$. \square

The last result of this section is an extension of [Sh2, Theorem 3.1].

Theorem 3.14. *Let $1 \leq p_0 < q_0 \leq \infty$. Suppose that T is a bounded sublinear operator on L^{p_0} . Assume that there exist constants $\alpha_2 > \alpha_1 > 1$, $C > 0$ such that*

$$\left(\int_B |Tf|^{q_0} \right)^{\frac{1}{q_0}} \leq C \left\{ \left(\int_{\alpha_1 B} |Tf|^{p_0} \right)^{\frac{1}{p_0}} + M(|f|^{p_0})^{\frac{1}{p_0}}(x) \right\}, \quad (3.13)$$

for all balls B , $x \in B$ and all $f \in L^\infty$ with compact support in $\mathbb{R}^n \setminus \alpha_2 B$. Let $p_0 < p < q_0$ and $w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{p})}'$. Then, there is a constant C such that

$$\|Tf\|_{L^p(w)} \leq C \|f\|_{L^p(w)}$$

for all $f \in L^\infty$ with compact support.

Proof. For any ball B , let $\mathcal{A}_B f = (1 - \chi_{\alpha_2 B}) f$. We fix $f \in L_c^\infty$, a ball B and $x \in B$. Using the L^{p_0} -boundedness of T , we have

$$\left(\int_{\alpha_1 B} |T(I - \mathcal{A}_B)f|^{p_0} \right)^{\frac{1}{p_0}} \leq C \left(\int_{\alpha_2 B} |f|^{p_0} \right)^{\frac{1}{p_0}} \leq C M(|f|^{p_0})^{\frac{1}{p_0}}(x). \quad (3.14)$$

In particular (3.7) holds since $\alpha_1 > 1$. Next, by (3.13) and since $|\mathcal{A}_B f| \leq |f|$ we have

$$\left(\int_B |T \mathcal{A}_B f|^{q_0} \right)^{\frac{1}{q_0}} \leq C \left\{ \left(\int_{\alpha_1 B} |T \mathcal{A}_B f|^{p_0} \right)^{\frac{1}{p_0}} + M(|f|^{p_0})^{\frac{1}{p_0}}(x) \right\}.$$

By (3.14) and the sublinearity of T , we obtain

$$\left(\int_B |T \mathcal{A}_B f|^{q_0} \right)^{\frac{1}{q_0}} \leq C M(|T f|^{p_0})^{\frac{1}{p_0}}(x) + C M(|f|^{p_0})^{\frac{1}{p_0}}(x),$$

which is (3.12) with $S_1 = I$ and $S_2 = 0$. We conclude on applying Theorem 3.13 with $\mathcal{D} = L_c^\infty$ and $\mathcal{E} = L^{p_0}$. \square

3.5. Commutators with BMO functions: part I. A slight strengthening of the hypotheses in Theorem 3.7 furnishes weighted L^p estimates for commutators with BMO functions.

Let $b \in \text{BMO}$ (BMO is for bounded mean oscillation), that is,

$$\|b\|_{\text{BMO}} = \sup_B \int_B |b(x) - b_B| dx < \infty,$$

where the supremum is taken over all balls and b_B stands for the average of b on B . Let T be a sublinear bounded operator on some L^{p_0} . Boundedness is assumed to avoid technical issues with the definition of the commutators. It could be relaxed, for instance, by imposing that T acts from $\mathcal{E} = \cap_p L_c^p$ into L^{p_0} . Sublinearity is defined in Section 3.4.

For any $k \in \mathbb{N}$ we define the k -th order commutator

$$T_b^k f(x) = T((b(x) - b)^k f)(x), \quad f \in L_c^\infty, \quad x \in \mathbb{R}^n.$$

Note that $T_b^0 = T$. Commutators are usually considered for linear operators T in which case they can be alternatively defined by recurrence: the first order commutator is

$$T_b^1 f(x) = [b, T]f(x) = b(x) T f(x) - T(b f)(x)$$

and for $k \geq 2$, the k -th order commutator is given by $T_b^k = [b, T_b^{k-1}]$.

We claim that since T is bounded in L^{p_0} then $T_b^k f$ is well defined in L_{loc}^q for any $0 < q < p_0$ and for any $f \in L_c^\infty$: take a cube Q containing the support of f and observe that by sublinearity for a.e. $x \in \mathbb{R}^n$

$$|T_b^k f(x)| \leq \sum_{m=0}^k C_{m,k} |b(x) - b_Q|^{k-m} |T((b - b_Q)^m f)(x)|.$$

John-Nirenberg's inequality implies

$$\int_Q |b(y) - b_Q|^{m p_0} |f(y)|^{p_0} dy \leq C \|f\|_{L^\infty} \|b\|_{\text{BMO}}^{m p_0} |Q| < +\infty.$$

Hence, $T((b - b_Q)^m f) \in L^{p_0}$ and the claim follows.

We are going to see that Theorem 3.1 can be applied to T_b^1 where the function H_2 involves $T = T_b^0$. The same will be done for T_b^k and in this case H_2 involves the preceding commutators $T, T_b^1, \dots, T_b^{k-1}$. Thus an induction argument (details are in Section 6.2) will lead us to the following estimates:

Theorem 3.15. *Let $1 \leq p_0 < q_0 \leq \infty$ and $k \in \mathbb{N}$. Suppose that T is a sublinear operator bounded on L^{p_0} , and that $\{\mathcal{A}_r\}_{r>0}$ is a family of operators acting from L_c^∞ into L^{p_0} . Assume that*

$$\left(\int_B |T(I - \mathcal{A}_{r(B)})f|^{p_0} \right)^{\frac{1}{p_0}} \leq C \sum_{j=1}^{\infty} \alpha_j \left(\int_{2^{j+1}B} |f|^{p_0} \right)^{\frac{1}{p_0}}, \quad (3.15)$$

and

$$\left(\int_B |T\mathcal{A}_{r(B)}f|^{q_0} \right)^{\frac{1}{q_0}} \leq \sum_{j=1}^{\infty} \alpha_j \left(\int_{2^{j+1}B} |Tf|^{p_0} \right)^{\frac{1}{p_0}}, \quad (3.16)$$

for all $f \in L_c^\infty$ and all ball B where $r(B)$ denotes its radius. Let $p_0 < p < q_0$ and $w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{p})}'$. If $\sum_j \alpha_j j^k < \infty$ then there is a constant C such that for all $f \in L_c^\infty$ and all $b \in \text{BMO}$,

$$\|T_b^k f\|_{L^p(w)} \leq C \|b\|_{\text{BMO}}^k \|f\|_{L^p(w)}, \quad (3.17)$$

for all $f \in L_c^\infty$.

Remark 3.16. Under the assumptions above, we have $\sum_j \alpha_j < \infty$ and so (3.15) and (3.16) imply respectively (3.7) and (3.8). Consequently, Theorem 3.7 applies to $T = T_b^0$ and yields its $L^p(w)$ -boundedness.

Observe that Remarks 3.9 and 3.10 apply to this result. Also, the operator T_b^k satisfies the vector-valued inequalities (3.10). The assumptions (3.15) and (3.16) can be relaxed in the spirit of Theorem 3.13 by allowing error terms in the right hand sides: details and proof are left to the interested reader.

Remark 3.17. As in [PT] one can linearize the k -th order commutator and consider the following multilinear commutators

$$T_{\vec{b}} f(x) = T\left(\left(\prod_{j=1}^k (b_j(x) - b_j)\right) f\right)(x).$$

where $\vec{b} = \{b_1, \dots, b_k\}$ is a family of BMO functions. Notice that if $b_1 = \dots = b_k = b$ we have that $T_{\vec{b}} = T_b^k$. The proof of Theorem 3.15 can be adapted to $T_{\vec{b}}$ and thus get the corresponding weighted estimates for it (see Remark 6.2). The precise statement is left to the reader.

4. THE SETS $\mathcal{W}_w(p_0, q_0)$ AND EXTRAPOLATION

4.1. The sets $\mathcal{W}_w(p_0, q_0)$. The conclusion of Theorem 3.7 with $S = I$ and $\mathcal{D} = L^{p_0}$ (and also of Theorems 3.13 and 3.15) can be rewritten as follows: given $w \in A_\infty$, we introduce the set

$$\mathcal{W}_w(p_0, q_0) = \left\{ p : p_0 < p < q_0, w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{p})}' \right\},$$

and we have shown that T is bounded on $L^p(w)$ whenever $p \in \mathcal{W}_w(p_0, q_0)$. Let us give some properties of this set.

Lemma 4.1. *Let $w \in A_\infty$ and $1 \leq p_0 < q_0 \leq \infty$. Then $\mathcal{W}_w(p_0, q_0) = (p_0 r_w, \frac{q_0}{(s_w)'})$ where*

$$r_w = \inf\{r \geq 1 : w \in A_r\}, \quad s_w = \sup\{s > 1 : w \in RH_s\}.$$

If $q_0 = \infty$, this result has to be understood in the following way: the set $\mathcal{W}_w(p_0, q_0)$ is defined by the only assumption $w \in A_{p/p_0}$ and the conclusion is $\mathcal{W}_w = (p_0 r_w, \infty)$.

Remark 4.2. Observe that if $1 \leq p_1 \leq p_0 \leq q_0 \leq q_1 \leq \infty$ then

$$\mathcal{W}_w(p_0, q_0) \subset \mathcal{W}_w(p_1, q_1) \subset \mathcal{W}_w(1, \infty) = (r_w, \infty) = \{1 < p < \infty : w \in A_p\}.$$

Remark 4.3. The set $\mathcal{W}_w(p_0, q_0)$ can be empty: indeed, for every $1 \leq p_0 < q_0 < \infty$, one can find $w \in A_\infty$ such that $\mathcal{W}_w(p_0, q_0) = \emptyset$. A very simple example in \mathbb{R} consists in taking $w(x) = |x|^\alpha$ for $\alpha = q_0/p_0 - 1$. Note that $w \in A_p$, $p > 1$, if and only if $\alpha < p - 1$ that is $p > \alpha + 1$ and so $r_w = \alpha + 1$. On the other hand, $w \in RH_\infty$ and so $s_w = \infty$. Therefore, $\mathcal{W}_w(p_0, q_0) = (p_0(1 + \alpha), q_0) = (q_0, q_0) = \emptyset$.

Proof of Lemma 4.1. We do the case $q_0 < \infty$, leaving the other one to the reader. If $p > p_0 r_w$ then $p/p_0 > r_w$ and so $w \in A_{p/p_0}$. If, additionally, $p < q_0/(s_w)'$ then $(q_0/p)' < s_w$ and so $w \in RH_{(q_0/p)'}$. Therefore we have shown that $(p_0 r_w, q_0/(s_w)') \subset \mathcal{W}_w(p_0, q_0)$.

To prove the converse, we observe that, by (iii) in Proposition 2.1, if $w \in A_{r_w}$ then $r_w = 1$: if $w \in A_{r_w}$ for $r_w > 1$, we have $w \in A_r$ for some $1 < r < r_w$ which contradicts the definition of r_w . In the same way, but this time by (iv) in Proposition 2.1, if $w \in RH_{s_w}$ then $s_w = \infty$.

Let $p \in \mathcal{W}_w(p_0, q_0)$. Since $w \in A_{p/p_0}$ then $r_w \leq p/p_0$. Besides, $r_w \neq p/p_0$ since $p/p_0 > 1$ and so $p > p_0 r_w$. On the other hand, $w \in RH_{(q_0/p)'}$ yields that $s_w \geq (q_0/p)'$. Besides, $s_w \neq (q_0/p)'$ since $q_0/p > 1$. This gives $p < q_0/(s_w)'$ as desired. \square

The duality for these classes goes as follows:

Lemma 4.4. *Given $p_0 < p < q_0$, we have*

$$w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'} \iff w^{1-p'} \in A_{\frac{p'}{(q_0)'}} \cap RH_{\left(\frac{(p_0)'}{p'}\right)'}$$

In other words, $p \in \mathcal{W}_w(p_0, q_0)$ if and only if $p' \in \mathcal{W}_{w^{1-p'}}((q_0)', (p_0)')$.

Proof. Set $q = \left(\frac{q_0}{p}\right)' \left(\frac{p}{p_0} - 1\right) + 1$. Using (vi) and (vii) in Proposition 2.1 we have

$$w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'} \iff w^{\left(\frac{q_0}{p}\right)'} \in A_{\left(\frac{q_0}{p}\right)' \left(\frac{p}{p_0} - 1\right) + 1} = A_q \iff w^{\left(\frac{q_0}{p}\right)'(1-q')} \in A_{q'}$$

and

$$w^{1-p'} \in A_{\frac{p'}{(q_0)'}} \cap RH_{\left(\frac{(p_0)'}{p'}\right)'} \iff w^{(1-p') \left(\frac{(p_0)'}{p'}\right)'} \in A_{\left(\frac{(p_0)'}{p'}\right)' \left(\frac{p'}{(q_0)'} - 1\right) + 1}.$$

Direct computations show

$$\left(\frac{q_0}{p}\right)'(1-q') = (1-p') \left(\frac{(p_0)'}{p'}\right)' \quad \text{and} \quad q' = \left(\frac{(p_0)'}{p'}\right)' \left(\frac{p'}{(q_0)'} - 1\right) + 1.$$

\square

Remark 4.5. Fix $1 < p < \infty$. Observe that if w is any given weight so that $w, w^{1-p'} \in L_{\text{loc}}^1$, then a given linear operator T is bounded on $L^p(w)$ if and only if its adjoint (with respect to dx) T^* is bounded on $L^{p'}(w^{1-p'})$. Therefore,

$$T : L^p(w) \longrightarrow L^p(w), \quad \text{for all } w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'}$$

if and only if

$$T^* : L^{p'}(w) \longrightarrow L^{p'}(w), \quad \text{for all } w \in A_{\frac{p'}{(q_0)'}} \cap RH_{\left(\frac{(p_0)'}{p'}\right)'}$$

We finish this section by giving families of weights on which r_w and s_w can be easily computed.

Lemma 4.6. *Let $f, g \in L^1(\mathbb{R}^n)$ be nontrivial functions, $r \geq 1$ and $1 < s \leq \infty$. Then:*

- (i) *Let $w = (Mf)^{-(r-1)}$ then $r_w = r$ and $s_w = \infty$, that is, $w \in A_p \cap RH_\infty$ for all $p > r$ (and $p = r$ if $r = 1$).*
- (ii) *Let $w = (Mf)^{1/s}$ then $r_w = 1$ and $s_w = s$, that is, $w \in A_1 \cap RH_q$ for all $q < s$ (and $q = s$ if $s = \infty$).*
- (iii) *If $w = (Mf)^{-(r-1)} + (Mg)^{1/s}$ then $w \in A_p \cap RH_q$ for all $p > r$ and $q < s$ (and $p = r$ if $r = 1$ and $q = s$ if $s = \infty$). Thus, $r_w \leq r$ and $s_w \geq s$.*

Proof. The cases $r = 1$ or $s = \infty$ are trivial. Given a nontrivial function $f \in L^1(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}$ we write $v_\alpha = (Mf)^\alpha$. If $\alpha = 0$ then $v_\alpha = 1 \in A_1 \cap RH_\infty$. If $0 < \alpha < 1$ then $v_\alpha \in A_1$ (see for instance [GR]). If $\alpha < 0$ then we see that $v_\alpha \in RH_\infty$: for a.e. $x \in B$

$$Mf(x)^\alpha = (Mf(x)^{\frac{1}{2}})^{2\alpha} \lesssim \left(\int_B (Mf)^{1/2} \right)^{2\alpha} \leq \int_B (Mf)^\alpha,$$

where we have used that $(Mf)^{1/2} \in A_1$ and also Jensen's inequality for the convex function $t \mapsto t^{2\alpha}$. Finally, it is easy to show that $v_\alpha \notin A_\infty$ for $\alpha \geq 1$. Indeed, assume that $v_\alpha = (Mf)^\alpha \in A_p$ for some $1 \leq p < \infty$. Then, $v_1 = v_\alpha^{1/\alpha} = Mf \in A_p$ as $\alpha \geq 1$. By (vi) in Proposition 2.1 we have that $v_1^{1-p'} \in A_{p'}$ and thus M is bounded on $L^{p'}(v_1^{1-p'})$. Applying this estimate to $f \in L^{p'}(v_1^{1-p'})$ (as $f \in L^1(\mathbb{R}^n)$) we obtain that $Mf \in L^1(\mathbb{R}^n)$ which only happens when $f \equiv 0$. This leads us to a contradiction since we have assumed that f is nontrivial.

We turn to showing (i). As $w = v_{-(r-1)}$, then $w \in RH_\infty$. Next, given $p > r$ the number $\alpha = (r-1)/(p-1)$ satisfies $0 < \alpha < 1$ and thus $v_\alpha \in A_1$. Notice that $w = 1 \cdot v_\alpha^{1-p} \in A_p$ (here we are using the “easy” part of the factorization of weights: if $w_1, w_2 \in A_1$ then $w_1 w_2^{1-p} \in A_p$). This shows that $w \in A_p$ for all $p > r$ and then $r_w \leq r$. To conclude we observe that $r_w = r$ as $w \notin A_r$: otherwise we would have $w^{1-r'} = Mf \in A_{r'}$ which cannot be the case as seen above.

We now consider (ii). Notice that $w = v_{1/s}$ with $1 < s < \infty$ and thus $w \in A_1$. Given $1 < q < s$, we see that $w \in RH_q$. Note that $w^q = v_{q/s} \in A_1$ as $q/s < 1$. Then, by (vii) in Proposition 2.1 it follows that $w \in RH_q \cap A_1$. Next, $w \notin RH_s$. If it were, then $w \in RH_{s+\varepsilon}$ for some $\varepsilon > 0$ and in particular $w^s = Mf \in A_\infty$ which is not true. Hence, $s_w = s$.

Note that (iii) follows from (i) and (ii) as $w = w_1 + w_2$ where $w_1 = (Mf)^{-(r-1)} \in A_p \cap RH_\infty$ and $w_2 = (Mf)^{1/s} \in A_1 \cap RH_q$ and $p > r$, $s < s_w$. \square

Remark 4.7. There are examples of functions f, g for which in (iii) we have $r_w < r$ and/or $s_w > s$. For instance, if $f = g = \chi_{B_0}$ with $B_0 = B(0, 1)$ then we have $Mf(x) \approx (1 + |x|)^{-n}$ and thus

$$w(x) \approx (1 + |x|)^{n(r-1)} \approx Mf(x)^{-(r-1)}.$$

Then, $r_w = r$ and $s_w = \infty$ (no matter the value of s). Similar examples can be given in the other direction.

Remark 4.8. The limit case in the latter result consists of taking f a Dirac mass at some given point x_0 , say $x_0 = 0$ for simplicity. In this case $Mf(x) = c|x|^{-n}$ is a power weight. In (i), (ii) and (iii) we respectively have $w_1(x) = c|x|^{n(r-1)}$, $w_2(x) = c|x|^{-n/s}$. Notice that $w_1 \notin A_r$, as $w_1^{1-r'} \notin L_{\text{loc}}^1(\mathbb{R}^n)$. Also, $w^s \notin RH_s$ as $w^s \notin L_{\text{loc}}^1(\mathbb{R}^n)$.

4.2. Extrapolation. Rubio de Francia's extrapolation theorem is a very powerful tool in Harmonic Analysis, see [Rub] and [Gar]: if some given operator T is bounded on $L^{p_0}(w)$ for every $w \in A_{p_0}$ and some $1 \leq p_0 < \infty$, then it is bounded on $L^p(w)$ for all $1 < p < \infty$ and all $w \in A_p$. So, the weighted norm inequality for one single exponent propagates to the whole range $(1, \infty)$. Notice that in our case the natural range of exponents is no longer $(1, \infty)$ but $(p_0, q_0) \subset (1, \infty)$.

Here we extend Rubio de Francia's result, showing that there is an extrapolation theorem adapted to the interval (p_0, q_0) which involves the classes of weights $A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{p})}'$. To state such result we first make some reductions. As it was observed in [CMP] (see also [CGMP]), one does not need to work with specific operator(s) since nothing about the operators themselves is used (like linearity or sublinearity) and they play no role. In other words, extrapolation is something about weights and pairs of functions. This point of view is very useful, for instance, when one tries to prove vector-valued inequalities since, as we see below, they follow at once from the corresponding scalar estimates.

So, sticking to the notation in [CMP], \mathcal{F} denotes a family of ordered pairs of non-negative, measurable functions (f, g) . In what follows, anytime we state an estimate

$$\|f\|_{L^p(w)} \leq C \|g\|_{L^p(w)}, \quad (f, g) \in \mathcal{F},$$

we mean that it holds for all $(f, g) \in \mathcal{F}$ for which the left-hand side is finite. The same is assumed when $L^{p, \infty}$ is written in place of L^p in the left hand side.

We can state our extrapolation result.

Theorem 4.9. *Let $0 < p_0 < q_0 \leq \infty$. Suppose that there exists p with $p_0 \leq p \leq q_0$, and $p < \infty$ if $q_0 = \infty$, such that for $(f, g) \in \mathcal{F}$,*

$$\|f\|_{L^p(w)} \leq C \|g\|_{L^p(w)}, \quad \text{for all } w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{p})}'. \quad (4.1)$$

Then, for all $p_0 < q < q_0$ and $(f, g) \in \mathcal{F}$ we have

$$\|f\|_{L^q(w)} \leq C \|g\|_{L^q(w)}, \quad \text{for all } w \in A_{\frac{q}{p_0}} \cap RH_{(\frac{q_0}{q})}'. \quad (4.2)$$

Moreover, for all $p_0 < q, r < q_0$ and $\{(f_j, g_j)\} \subset \mathcal{F}$ we have

$$\left\| \left(\sum_j (f_j)^r \right)^{1/r} \right\|_{L^q(w)} \leq C \left\| \left(\sum_j (g_j)^r \right)^{1/r} \right\|_{L^q(w)}, \quad \text{for all } w \in A_{\frac{q}{p_0}} \cap RH_{(\frac{q_0}{q})}'. \quad (4.3)$$

The proof of this result is in Section 6.3. As an immediate consequence we can also extrapolate from weak-type estimates:

Corollary 4.10. *Let $0 < p_0 < q_0 \leq \infty$. Suppose that there exists p with $p_0 \leq p \leq q_0$, and $p < \infty$ if $q_0 = \infty$, such that for $(f, g) \in \mathcal{F}$,*

$$\|f\|_{L^{p, \infty}(w)} \leq C \|g\|_{L^p(w)} \quad \text{for all } w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{p})}'. \quad (4.4)$$

Then, for all $p_0 < q < q_0$ and $(f, g) \in \mathcal{F}$ we have

$$\|f\|_{L^{q,\infty}(w)} \leq C \|g\|_{L^q(w)} \quad \text{for all } w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{q})}'. \quad (4.5)$$

Proof. We follow the simple method used in [GM], for which the point of view of pairs of functions is particularly useful. Given $(f, g) \in \mathcal{F}$ and any $\lambda > 0$ we define a new pair of functions (f_λ, g) where $f_\lambda = \lambda \chi_{E_\lambda(f)}$ and $E_\lambda(f) = \{f > \lambda\}$. Thus (4.4) implies

$$\|f_\lambda\|_{L^p(w)} = \lambda w(E_\lambda(f))^{\frac{1}{p}} \leq \sup_\lambda \lambda w(E_\lambda(f))^{\frac{1}{p}} = \|f\|_{L^{p,\infty}(w)} \leq C \|g\|_{L^p(w)}$$

for all $w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{p})}'$. Applying Theorem 4.9, the family $\tilde{\mathcal{F}}$ of pairs (f_λ, g) satisfy (4.2) with C independent of λ , and taking the supremum on $\lambda > 0$ we obtain (4.5). \square

Remark 4.11. Define the following sets, given an operator T defined at least on $C_0^\infty(\mathbb{R}^n)$:

$$\mathcal{W}(T) = \{(p, w) \in (1, \infty) \times A_\infty : \|Tf\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}\};$$

for $1 < p < \infty$, $\mathcal{W}^p(T) = \{w \in A_\infty : (p, w) \in \mathcal{W}(T)\}$; and for $w \in A_\infty$, $\mathcal{W}_w(T) = \{p \in (1, \infty) : (p, w) \in \mathcal{W}(T)\}$.

Next define for $1 \leq p_0 < q_0 \leq \infty$

$$\mathcal{W}(p_0, q_0) = \{(p, w) \in (p_0, q_0) \times A_\infty : w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{p})}'\};$$

for $p_0 < p < q_0$, $\mathcal{W}^p(p_0, q_0) = \{w \in A_\infty : (p, w) \in \mathcal{W}(p_0, q_0)\}$ and for $w \in A_\infty$, $\mathcal{W}_w(p_0, q_0) = \{p \in (p_0, q_0) : (p, w) \in \mathcal{W}(p_0, q_0)\}$. Recall that the smallest p_0 (resp. the largest q_0), the largest the class $\mathcal{W}(p_0, q_0)$.

For example, if T is a Calderón-Zygmund operator, then $\mathcal{W}(T)$ contains the largest of all classes, namely $\mathcal{W}(1, \infty)$ and this is optimal. Theorem 3.7 (with $S = I$ and $\mathcal{D} = L^{p_0}$) provides us with a sufficient condition on T to obtain that $\mathcal{W}(p_0, q_0) \subset \mathcal{W}(T)$.

Our extrapolation result shows that, given T and p , if some $\mathcal{W}^p(p_0, q_0)$ is contained in $\mathcal{W}^p(T)$ then for all $q \in (p_0, q_0)$, $\mathcal{W}^q(p_0, q_0)$ is contained in $\mathcal{W}^q(T)$. In other words, $\mathcal{W}^p(p_0, q_0) \subset \mathcal{W}^p(T)$ for one p implies $\mathcal{W}(p_0, q_0) \subset \mathcal{W}(T)$. The class of weights $\mathcal{W}^p(p_0, q_0)$ is thus the natural one for weighted L^p boundedness within the range $p_0 < p < q_0$. However, the inclusion could be strict for a particular operator T as we will see in [AM4].

5. EXTENSION TO SPACES OF HOMOGENEOUS TYPE

In [AM3], we apply our results in \mathbb{R}^n equipped with the doubling measure $d\mu(x) = w(x)dx$ with $w \in A_\infty$ (in this case $w(\mathbb{R}^n) = \infty$). In [AM4], we change \mathbb{R}^n to a manifold or a Lie group. Hence, one needs to discuss the extension of our results to spaces of homogeneous type.

Let (\mathcal{X}, d, μ) be a space of homogeneous type, that is, a set \mathcal{X} endowed with a distance d (and even a quasi-distance) and a non-negative Borel measure μ on \mathcal{X} such that the doubling condition

$$\mu(B(x, 2r)) \leq C_0 \mu(B(x, r)) < \infty, \quad (5.1)$$

holds for all $x \in \mathcal{X}$ and $r > 0$, where $B(x, r) = \{y \in \mathcal{X} : d(x, y) < r\}$.

The results from Harmonic Analysis that we have used in Euclidean spaces remain true in this context (see for example [CW], [Chr], [Ste]). For instance, Vitali's

covering lemma, weak-type $(1, 1)$ hence strong-type (p, p) for $1 < p \leq \infty$ of the Hardy-Littlewood maximal function, Whitney's covering lemma ... The theory of Muckenhoupt weights runs parallel to the classical case and one may prove all the statements in Proposition 2.1 with the appropriate changes (see [ST, Chapter I]).

Hence, Theorems 3.1, 3.7, 3.13, 3.14, 3.15, 4.9 all have their counterpart in spaces of homogeneous type with almost identical proofs whenever $\mu(\mathcal{X}) = \infty$.

When $\mu(\mathcal{X}) < \infty$ (for example, \mathcal{X} is a bounded Lipschitz domain in \mathbb{R}^n) some adjustments are needed. In Theorem 3.1, assuming that $F \in L^1$ then the two parameter good- λ estimate (3.3) holds for $\lambda > \lambda_0 = C_0 \mu(\mathcal{X})^{-1} (\|F\|_{L^1} + \|H_1\|_{L^1})$. This condition guarantees that $\mu(E_\lambda) < \mu(\mathcal{X})$ and so $E_\lambda \subsetneq \mathcal{X}$. The Whitney covering argument can be performed and the proof presented above works in the same way. Thus, when proving the analog of (3.4), one has to split the integral in two parts: $\lambda \geq \lambda_0$ and $\lambda \leq \lambda_0$. For the first one, we use (3.3). The piece $\lambda \leq \lambda_0$ is estimated by observing that $w\{MF > \lambda\} \leq w(\mathcal{X}) < \infty$ (since $\mu(\mathcal{X}) < \infty$ if and only if \mathcal{X} is bounded, see for instance [Ma2]). Thus, it can be proved that

$$\|MF\|_{L^p(w)} \leq C (\|G\|_{L^p(w)} + \|MH_1\|_{L^p(w)} + \|H_2\|_{L^p(w)} + \|F\|_{L^1(\mu)} + \|MH_1\|_{L^1(\mu)}).$$

The same occurs with the estimates in $L^{p,\infty}(w)$.

The latter inequality allows one to obtain Theorem 3.7 assuming further that T is bounded on L^{p_0} (this happens all the time in applications, see [AM3]). The only change is for the term $\|F\|_{L^1}$ where $F = |Tf|^{p_0}$ (notice that $H_1 = H_2 = 0$ in this case):

$$\|F\|_{L^1(\mu)} = \|Tf\|_{L^{p_0}(\mu)}^{p_0} \lesssim \|f\|_{L^{p_0}(\mu)}^{p_0} \leq \|f\|_{L^p(w)}^{p_0} \int_{\mathcal{X}} w^{1-(p/p_0)'} d\mu \lesssim \|f\|_{L^p(w)}^{p_0}.$$

For the last inequality, we observe that since $w \in A_{(p/p_0)'}$, then $w^{1-(p/p_0)'} \in A_{(p/p_0)'}$ and so it is a doubling measure which implies as noted before that $w^{1-(p/p_0)'}(\mathcal{X}) < \infty$ as \mathcal{X} is bounded. Similar modifications can be carried out with Theorems 3.13 and 3.14. Precise statements and details of proofs are left to the interested reader.

6. PROOFS OF THE MAIN RESULTS

We prove Theorem 3.1, Theorem 3.15, Theorem 4.9.

6.1. Proof of Theorem 3.1. The proof follows the ideas in [Au1]. It suffices to consider the case $H_2 = G$: indeed, set $\tilde{G} = G + H_2$. Then (3.1) holds with \tilde{G} in place of H_2 and also (3.2) holds with \tilde{G} in place of G .

So from now on we assume that $H_2 = G$. Set $E_\lambda = \{MF + MH_1 > \lambda\}$ which is assumed to have finite measure (otherwise there is nothing to prove). As M is the uncentered maximal function (over cubes instead of balls), E_λ is an open set. Hence, Whitney's decomposition gives us a family of pairwise disjoint cubes $\{Q_j\}_j$ so that $E_\lambda = \cup_j Q_j$ and with the property that $4Q_j$ meets E_λ^c , that is, there exists $x_j \in 4Q_j$ such that

$$MF(x_j) + MH_1(x_j) \leq \lambda.$$

Set $B_\lambda = \{MF > K\lambda, 2G \leq \gamma\lambda\}$. Since $K \geq 1$ we have that $B_\lambda \subset E_\lambda$. Therefore $B_\lambda \subset \cup_j B_\lambda \cap Q_j$. For each j we assume that $B_\lambda \cap Q_j \neq \emptyset$ (otherwise we discard this

cube) and so there is $\bar{x}_j \in Q_j$ so that $G(\bar{x}_j) \leq \gamma \lambda/2$. Since $MF(x_j) \leq \lambda$, there is C_0 depending only on dimension such that for every $K \geq C_0$ we have

$$\begin{aligned} |B_\lambda \cap Q_j| &\leq |\{MF > K\lambda\} \cap Q_j| \leq |\{M(F\chi_{8Q_j}) > (K/C_0)\lambda\}| \\ &\leq |\{M(G_{8Q_j}\chi_{8Q_j}) > (K/2C_0)\lambda\}| + |\{M(H_{8Q_j}\chi_{8Q_j}) > (K/2C_0)\lambda\}|, \end{aligned}$$

where we have used $F\chi_{8Q_j} \leq G_{8Q_j}\chi_{8Q_j} + H_{8Q_j}\chi_{8Q_j}$ a.e. and χ_{8Q_j} is the indicator function of $8Q_j$. Let c_p be the weak-type (p, p) bound of the maximal function. By (3.2) and $\bar{x}_j \in Q_j \subset 8Q_j$, we obtain

$$\begin{aligned} |\{M(G_{8Q_j}\chi_{8Q_j}) > (K/2C_0)\lambda\}| &\leq \frac{2C_0c_1}{K\lambda} \int_{8Q_j} G_{8Q_j} \leq \frac{2C_0c_1}{K\lambda} |8Q_j| G(\bar{x}_j) \\ &\leq \frac{8^n C_0 c_1}{K} |Q_j| \gamma. \end{aligned}$$

Next, assume first that $q < \infty$. By (3.1) and $x_j, \bar{x}_j \in 8Q_j$, we obtain

$$\begin{aligned} |\{M(H_{8Q_j}\chi_{8Q_j}) > (K/2C_0)\lambda\}| &\leq \left(\frac{2C_0c_q}{K\lambda}\right)^q \int_{8Q_j} H_{8Q_j}^q \\ &\leq \left(\frac{2C_0c_q}{K\lambda}\right)^q |8Q_j| a^q (MF(x_j) + MH_1(x_j) + G(\bar{x}_j))^q \leq \frac{(4C_0c_qa)^q 8^n}{K^q} |Q_j|. \end{aligned}$$

These two estimates yield

$$|B_\lambda \cap Q_j| \leq C \left(\frac{a^q}{K^q} + \frac{\gamma}{K} \right) |Q_j|.$$

At this point, we use that $w \in RH_{s'}$. If $s' < \infty$, for any cube Q and any measurable set $E \subset Q$ we have

$$\frac{w(E)}{w(Q)} \leq \frac{|Q|}{w(Q)} \left(\int_Q w^{s'} \right)^{\frac{1}{s'}} \left(\frac{|E|}{|Q|} \right)^{\frac{1}{s}} \leq C_w \left(\frac{|E|}{|Q|} \right)^{\frac{1}{s}}.$$

Note that the same conclusion holds in the case $s' = \infty$. Applying this to $B_\lambda \cap Q_j \subset Q_j$ we have

$$w(B_\lambda \cap Q_j) \leq C_w C \left(\frac{a^q}{K^q} + \frac{\gamma}{K} \right)^{\frac{1}{s}} w(Q_j).$$

Hence, using that the Whitney cubes are disjoint we have

$$w(B_\lambda) \leq \sum_j w(B_\lambda \cap Q_j) \leq C \left(\frac{a^q}{K^q} + \frac{\gamma}{K} \right)^{\frac{1}{s}} \sum_j w(Q_j) = C \left(\frac{a^q}{K^q} + \frac{\gamma}{K} \right)^{\frac{1}{s}} w(E_\lambda)$$

which is (3.3).

When $q = \infty$, then by (3.1)

$$\|M(H_{8Q_j}\chi_{8Q_j})\|_{L^\infty} \leq \|H_{8Q_j}\chi_{8Q_j}\|_{L^\infty} \leq a(MF(x_j) + MH_1(x_j) + G(\bar{x}_j)) \leq 2a\lambda.$$

Thus choosing $K \geq 4aC_0$ it follows that $\{M(H_{8Q_j}\chi_{8Q_j}) > (K/2C_0)\lambda\} = \emptyset$. Proceeding as before, we get the desired estimate (with $K^{-q} = 0$).

Next we show (3.4) when it is assumed that $MF \in L^p(w)$. Integrating the two-parameter good- λ inequality (3.3) against $p \lambda^{p-1} d\lambda$ on $(0, \infty)$, for $0 < p < \infty$,

$$\|MF\|_{L^p(w)}^p \leq C K^p \left(\frac{a^q}{K^q} + \frac{\gamma}{K} \right)^{\frac{1}{s}} (\|MF\|_{L^p(w)}^p + \|MH_1\|_{L^p(w)}^p) + \frac{2^p K^p}{\gamma^p} \|G\|_{L^p(w)}^p.$$

Thus, as $\|MF\|_{L^p(w)} < \infty$, for $0 < p < \frac{q}{s}$ we can choose K large enough and then γ small enough so that the constant in front of the first term in the right-hand side is smaller than $\frac{1}{2}$, leading us to (3.4). In the same way, but this time assuming that $MF \in L^{p,\infty}(w)$, one shows the corresponding estimate in $L^{p,\infty}(w)$.

Observe that in the case $q = \infty$, K is already chosen and we only have to take some small γ . Thus, the corresponding estimates holds for $0 < p < \infty$ no matter the value of s .

Now, we consider the case $p \geq 1$ and $F \in L^1$. We assume that the right-hand side of (3.4) is finite, otherwise there is nothing to prove. It suffices to consider the case $w \in L^\infty$: indeed we can take $w_N = \min\{w, N\}$ with $N > 0$. As $w \in RH_{s'}$ then $w_N \in RH_{s'}$ with constant that is uniformly controlled in N . Notice that if we show (3.4) with w_N and with constants that do not depend on N , by taking limits as $N \rightarrow \infty$, we conclude the desired estimate with w .

So we assume that $w \in L^\infty$. Let f be the non-negative function defined by $f(\lambda) = p\lambda^p w\{MF > \lambda\}$, $\lambda > 0$. Notice that for any $0 < \lambda_0 < \lambda_1 < \infty$, $\int_{\lambda_0}^{\lambda_1} f(\lambda) \frac{d\lambda}{\lambda}$ exists and is finite. By (3.3) we have

$$\begin{aligned} \int_{\lambda_0}^{\lambda_1} f(\lambda) \frac{d\lambda}{\lambda} &= \int_{\frac{\lambda_0}{K}}^{\frac{\lambda_1}{K}} f(K\lambda) \frac{d\lambda}{\lambda} \\ &\leq C K^p 2^p \left(\frac{a^q}{K^q} + \frac{\gamma}{K} \right)^{\frac{1}{s}} \left(\int_{\frac{\lambda_0}{2K}}^{\frac{\lambda_1}{2K}} f(\lambda) \frac{d\lambda}{\lambda} + \|MH_1\|_{L^p(w)}^p \right) + \frac{2^p K^p}{\gamma^p} \|G\|_{L^p(w)}^p \\ &\leq \frac{1}{2} \int_{\frac{\lambda_0}{2K}}^{\frac{\lambda_1}{K}} f(\lambda) \frac{d\lambda}{\lambda} + R \end{aligned}$$

where in the last inequality we have picked K large enough and then γ small enough so that the constant in front of the first term in the right-hand side is smaller than $1/2$. Also we have written R for the remainder terms, that is, $R = C (\|MH_1\|_{L^p(w)}^p + \|G\|_{L^p(w)}^p) < \infty$. We take $\lambda_0 = K^{-n}$ and $\lambda_1 = K^m$ with $n, m \geq 1$ and so

$$\begin{aligned} \int_{K^{-n}}^{K^{m-1}} f(\lambda) \frac{d\lambda}{\lambda} &\leq \int_{K^{-n}}^{K^m} f(\lambda) \frac{d\lambda}{\lambda} \leq \frac{1}{2} \int_{\frac{K^{-n-1}}{2}}^{K^{m-1}} f(\lambda) \frac{d\lambda}{\lambda} + R \\ &\leq \frac{1}{2} \int_{K^{-n}}^{K^{m-1}} f(\lambda) \frac{d\lambda}{\lambda} + \frac{1}{2} \int_{\frac{K^{-n-1}}{2}}^{K^{-n}} f(\lambda) \frac{d\lambda}{\lambda} + R. \end{aligned}$$

Hence,

$$\int_{K^{-n}}^{K^{m-1}} f(\lambda) \frac{d\lambda}{\lambda} \leq \int_{\frac{K^{-n-1}}{2}}^{K^{-n}} f(\lambda) \frac{d\lambda}{\lambda} + 2R.$$

Since M is of weak-type $(1, 1)$, $w \in L^\infty$ and $K \geq 1$ we have

$$\int_{\frac{K-n-1}{2}}^{K-n} f(\lambda) \frac{d\lambda}{\lambda} \leq C \|w\|_{L^\infty} \|F\|_{L^1} \begin{cases} \log 2K & \text{if } p = 1, \\ 1 & \text{if } p > 1, \end{cases}$$

bound which does not depend on n . We conclude that

$$\|MF\|_{L^p(w)}^p = \int_0^\infty f(\lambda) \frac{d\lambda}{\lambda} = \lim_{n,m \rightarrow \infty} \int_{K^{-n}}^{K^{m-1}} f(\lambda) \frac{d\lambda}{\lambda} < \infty,$$

so that $MF \in L^p(w)$. Therefore, (3.4) holds with constants that do *not* depend on $\|w\|_{L^\infty}$. A very similar argument applies for the weak-type estimate. Details are left to the reader. \square

6.2. Proof of Theorem 3.15. Before starting the proof, let us introduce some notation (see [BS] for more details). Let ϕ be a Young function: $\phi : [0, \infty) \rightarrow [0, \infty)$ is continuous, convex, increasing and satisfies $\phi(0+) = 0$, $\phi(\infty) = \infty$. Given a cube Q we define the localized Luxemburg's norm

$$\|f\|_{\phi, Q} = \inf \left\{ \lambda > 0 : \int_Q \phi \left(\frac{|f|}{\lambda} \right) \leq 1 \right\},$$

and then the maximal operator

$$M_\phi f(x) = \sup_{Q \ni x} \|f\|_{\phi, Q}.$$

In the definition of $\|\cdot\|_{\phi, Q}$, if the probability measure $dx/|Q|$ is replaced by dx and Q by \mathbb{R}^n , then one has the Luxemburg's norm $\|\cdot\|_\phi$ which allows one to define the Orlicz space L^ϕ .

Some specific examples needed here are $\phi(t) \approx e^{t^r}$ for $t \geq 1$ which gives the classical space $\exp L^r$ and $\phi(t) = t(1 + \log^+ t)^\alpha$ with $\alpha > 0$ that gives the space $L(\log L)^\alpha$. In this latter case, it is well known that for $k \geq 1$, we have $M_{L(\log L)^{k-1}} f \approx M^k f$ where M^k is the k -iteration of M .

John-Nirenberg's inequality implies that for any function $b \in \text{BMO}$ and any cube Q we have $\|b - b_Q\|_{\exp L, Q} \lesssim \|b\|_{\text{BMO}}$. This yields the following estimates: First, for each cube Q and $x \in Q$

$$\begin{aligned} \int_Q |b - b_Q|^{k p_0} |f|^{p_0} &\leq \|b - b_Q\|_{\exp L, Q}^{k p_0} \| |f|^{p_0} \|_{L(\log L)^{k p_0}, Q} \\ &\lesssim \|b\|_{\text{BMO}}^{k p_0} M_{L(\log L)^{k p_0}} (|f|^{p_0})(x) \lesssim \|b\|_{\text{BMO}}^{k p_0} M^{[k p_0] + 2} (|f|^{p_0})(x), \end{aligned} \quad (6.1)$$

where $[s]$ is the integer part of s (if $k p_0 \in \mathbb{N}$, then one can take $M^{[k p_0] + 1}$). Second, for each $j \geq 1$ and each Q ,

$$\begin{aligned} \|b - b_{2^j Q}\|_{\exp L, 2^j Q} &\leq \|b - b_{2^j Q}\|_{\exp L, 2^j Q} + |b_{2^j Q} - b_{2Q}| \lesssim \|b\|_{\text{BMO}} + \sum_{l=1}^{j-1} |b_{2^{l+1} Q} - b_{2^l Q}| \\ &\lesssim \|b\|_{\text{BMO}} + \sum_{l=1}^{j-1} \int_{2^{l+1} Q} |b - b_{2^{l+1} Q}| \lesssim j \|b\|_{\text{BMO}}. \end{aligned} \quad (6.2)$$

The following auxiliary result allows us to assume further that $b \in L^\infty$. The proof is postponed until the end of this section.

Lemma 6.1. *Let $1 \leq p_0 < p < \infty$, $k \in \mathbb{N}$ and $w \in A_\infty$. Let T be a sublinear operator bounded on L^{p_0} .*

- (i) *If $b \in \text{BMO} \cap L^\infty$ and $f \in L_c^\infty$, then $T_b^k f \in L^{p_0}$.*
- (ii) *Assume that for any $b \in \text{BMO} \cap L^\infty$ and for any $f \in L_c^\infty$ we have that*

$$\|T_b^k f\|_{L^p(w)} \leq C_0 \|b\|_{\text{BMO}}^k \|f\|_{L^p(w)}, \quad (6.3)$$

where C_0 does not depend on b and f . Then for all $b \in \text{BMO}$, (6.3) holds with constant $2^k C_0$ instead of C_0 .

Part (ii) in this latter result ensures that it suffices to consider the case $b \in L^\infty$ (provided the constants obtained do not depend on b). So from now on we assume that $b \in L^\infty$ and obtain (6.3) with C_0 independent of b and f . Note that by homogeneity we can also assume that $\|b\|_{\text{BMO}} = 1$.

We proceed by induction. As mentioned in Remark 3.16, the case $k = 0$ follows from Theorem 3.7. We write the case $k = 1$ in full detail and indicate how to pass from $k - 1$ to k as the argument is essentially the same. Let us fix $p_0 < p < q_0$ and $w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'}'$. We assume that $q_0 < \infty$, for $q_0 = \infty$ the main ideas are the same and details are left to the interested reader.

Case $k = 1$: We combine the ideas in the proof of Theorem 3.7 with techniques for commutators, see [Per]. Let $f \in L_c^\infty$ and set $F = |T_b^1 f|^{p_0}$. Note that $F \in L^1$ by (i) in Lemma 6.1 (this is the only place in this step where we use that $b \in L^\infty$). Given a cube Q , we set $f_{Q,b} = (b_{4Q} - b)f$ and decompose T_b^1 as follows:

$$\begin{aligned} |T_b^1 f(x)| &= |T((b(x) - b)f)(x)| \leq |b(x) - b_{4Q}| |Tf(x)| + |T((b_{4Q} - b)f)(x)| \\ &\leq |b(x) - b_{4Q}| |Tf(x)| + |T(I - \mathcal{A}_{r(Q)})f_{Q,b}(x)| + |T\mathcal{A}_{r(Q)}f_{Q,b}(x)|. \end{aligned}$$

With the notation of Theorem 3.1, we observe that $F \leq G_Q + H_Q$ where

$$G_Q = 4^{p_0-1} (G_{Q,1} + G_{Q,2}) = 4^{p_0-1} (|b - b_{4Q}|^{p_0} |Tf|^{p_0} + |T(I - \mathcal{A}_{r(Q)})f_{Q,b}|^{p_0})$$

and $H_Q = 2^{p_0-1} |T\mathcal{A}_{r(Q)}f_{Q,b}|^{p_0}$.

We first estimate the average of G_Q on Q . Fix any $x \in Q$. By (6.1) with $k = 1$,

$$\int_Q G_{Q,1} = \int_Q |b - b_{4Q}|^{p_0} |Tf|^{p_0} \lesssim \|b\|_{\text{BMO}}^{p_0} M^{[p_0]+2}(|Tf|^{p_0})(x).$$

Using (3.15), (6.1) and (6.2),

$$\begin{aligned} \left(\int_Q G_{Q,2} \right)^{\frac{1}{p_0}} &= \left(\int_Q |T(I - \mathcal{A}_{r(Q)})f_{Q,b}|^{p_0} \right)^{\frac{1}{p_0}} \lesssim \sum_{j=1}^{\infty} \alpha_j \left(\int_{2^{j+1}Q} |f_{Q,b}|^{p_0} \right)^{\frac{1}{p_0}} \\ &\leq \sum_{j=1}^{\infty} \alpha_j \|b - b_{4Q}\|_{\text{exp}L, 2^{j+1}Q} M^{[p_0]+2}(|f|^{p_0})^{\frac{1}{p_0}}(x) \\ &\lesssim \|b\|_{\text{BMO}} M^{[p_0]+2}(|f|^{p_0})(x)^{\frac{1}{p_0}} \sum_{j=1}^{\infty} \alpha_j j \lesssim M^{[p_0]+2}(|f|^{p_0})^{\frac{1}{p_0}}(x), \end{aligned}$$

since $\sum_j \alpha_j j < \infty$. Hence, for any $x \in Q$

$$\int_Q G_Q \leq C (M^{[p_0]+2}(|Tf|^{p_0})(x) + M^{[p_0]+2}(|f|^{p_0})(x)) \equiv G(x).$$

We next estimate the average of H_Q^q on Q with $q = q_0/p_0$. Using (3.16) and proceeding as before

$$\begin{aligned} \left(\int_Q H_Q^q \right)^{\frac{1}{q_0}} &= 2^{(p_0-1)/p_0} \left(\int_Q |T \mathcal{A}_{r(Q)} f_{Q,b}|^{q_0} \right)^{\frac{1}{q_0}} \lesssim \sum_{j=1}^{\infty} \alpha_j \left(\int_{2^{j+1}Q} |T f_{Q,b}|^{p_0} \right)^{\frac{1}{p_0}} \\ &\leq \sum_{j=1}^{\infty} \alpha_j \left(\int_{2^{j+1}Q} |T_b^1 f|^{p_0} \right)^{\frac{1}{p_0}} + \sum_{j \geq 1} \alpha_j \left(\int_{2^{j+1}Q} |b - b_{4Q}|^{p_0} |Tf|^{p_0} \right)^{\frac{1}{p_0}} \\ &\lesssim (MF)^{\frac{1}{p_0}}(x) + \sum_{j=1}^{\infty} \alpha_j \|b - b_{4Q}\|_{\exp L, 2^{j+1}Q} M^{[p_0]+2}(|Tf|^{p_0})^{\frac{1}{p_0}}(\bar{x}) \\ &\lesssim (MF)^{\frac{1}{p_0}}(x) + M^{[p_0]+2}(|Tf|^{p_0})^{\frac{1}{p_0}}(\bar{x}) \sum_{j=1}^{\infty} \alpha_j j \\ &\lesssim (MF)^{\frac{1}{p_0}}(x) + M^{[p_0]+2}(|Tf|^{p_0})^{\frac{1}{p_0}}(\bar{x}), \end{aligned}$$

for any $x, \bar{x} \in Q$, where we have used that $\sum_j \alpha_j j < \infty$. Thus we have obtained

$$\left(\int_Q H_Q^q \right)^{\frac{1}{q}} \leq C (MF(x) + M^{[p_0]+2}(|Tf|^{p_0})(\bar{x})) \equiv C (MF(x) + H_2(\bar{x})).$$

As mentioned before $F \in L^1$. Since $w \in RH_{(q_0/p)'}'$, applying Theorem 3.1 and Remark 3.6 (since $q_0 < \infty$ implies $q < \infty$) with p/p_0 in place of p and $s = q_0/p$, we obtain

$$\begin{aligned} \|T_b^1 f\|_{L^{p_0}(w)}^{p_0} &\leq \|MF\|_{L^{\frac{p}{p_0}}(w)} \lesssim \|G\|_{L^{\frac{p}{p_0}}(w)} + \|H_2\|_{L^{\frac{p}{p_0}}(w)} \\ &\lesssim \|M^{[p_0]+2}(|f|^{p_0})\|_{L^{\frac{p}{p_0}}(w)} + \|M^{[p_0]+2}(|Tf|^{p_0})\|_{L^{\frac{p}{p_0}}(w)} \\ &\lesssim \|f\|_{L^p(w)}^{p_0} + \|Tf\|_{L^p(w)}^{p_0} \lesssim \|f\|_{L^p(w)}^{p_0}, \end{aligned}$$

where we have used the boundedness of M (hence, M^2, M^3, \dots) on $L^{\frac{p}{p_0}}(w)$ as $w \in A_{p/p_0}$ with $p_0 < p$, and also Remark 3.16. Let us emphasize that none of the constants depend on b or f .

Case k : We now sketch the induction argument. Assume that we have already proved the cases $m = 0, \dots, k-1$. Let $f \in L_c^\infty$. Given a cube Q , write $f_{Q,b} = (b_{4Q} - b)^k f$ and decompose T_b^k as follows:

$$\begin{aligned} |T_b^k f(x)| &= |T((b(x) - b)^k f)(x)| \\ &\leq \sum_{m=0}^{k-1} C_{k,m} |b(x) - b_{4Q}|^{k-m} |T_b^m f(x)| + |T((b_{4Q} - b)^k f)(x)| \\ &\lesssim \sum_{m=0}^{k-1} |b(x) - b_{4Q}|^{k-m} |T_b^m f(x)| + |T(I - \mathcal{A}_{r(Q)})f_{Q,b}(x)| + |T \mathcal{A}_{r(Q)} f_{Q,b}(x)|. \end{aligned}$$

Following the notation of Theorem 3.1, we set $F = |T_b^k f|^{p_0} \in L^1$ by (i) in Lemma 6.1. Observe that $F \leq G_Q + H_Q$ where

$$G_Q = 4^{p_0-1} C \left(\left(\sum_{m=0}^{k-1} |b - b_{4Q}|^{k-m} |T_b^m f| \right)^{p_0} + |T(I - \mathcal{A}_{r(Q)})f_{Q,b}|^{p_0} \right)$$

and $H_Q = 2^{p_0-1} |T \mathcal{A}_{r(Q)} f_{Q,b}|^{p_0}$. Proceeding as before we obtain for any $x \in Q$

$$\int_Q G_Q \leq C \left(\sum_{m=0}^{k-1} M^{[(k-m)p_0]+2}(|T_b^m f|^{p_0})(x) + M^{[kp_0]+2}(|f|^{p_0})(x) \right) \equiv G(x),$$

and for $q = q_0/p_0$

$$\left(\int_Q H_Q^q \right)^{\frac{1}{q}} \leq C \left(MF(x) + \sum_{m=0}^{k-1} M^{[(k-m)p_0]+2}(|T_b^m f|^{p_0})(\bar{x}) \right) \equiv C \left(MF(x) + H_2(\bar{x}) \right).$$

Therefore, as $F \in L^1$, Theorem 3.1 gives us as before

$$\begin{aligned} \|T_b^k f\|_{L^{p_0}(w)}^{p_0} &\leq \|MF\|_{L^{\frac{p}{p_0}}(w)} \lesssim \|G\|_{L^{\frac{p}{p_0}}(w)} + \|H_2\|_{L^{\frac{p}{p_0}}(w)} \\ &\lesssim \|M^{[kp_0]+2}(|f|^{p_0})\|_{L^{\frac{p}{p_0}}(w)} + \sum_{m=0}^{k-1} \|M^{[(k-m)p_0]+2}(|T_b^m f|^{p_0})\|_{L^{\frac{p}{p_0}}(w)} \\ &\lesssim \|f\|_{L^{p_0}(w)}^{p_0} + \sum_{m=0}^{k-1} \|T_b^m f\|_{L^{p_0}(w)}^{p_0} \lesssim \|f\|_{L^p(w)}^{p_0}, \end{aligned}$$

where we have used the boundedness on $L^{\frac{p}{p_0}}(w)$ of the iterations of M (as $w \in A_{p/p_0}$ and $p > p_0$) and the induction hypothesis on T_b^m , $m = 0, \dots, k-1$. Let us point out again that none of the constants involved in the proof depend on b and f .

Proof of Lemma 6.1. Some of the ideas of the following argument are taken from [Per] where this is proved for Calderón-Zygmund operators. Note that there, one has size and smoothness estimates for the kernels and here such conditions are not assumed.

Fix $f \in L_c^\infty$. Note that (i) follows easily observing that

$$|T_b^k f(x)| \lesssim \sum_{m=0}^k |b(x)|^{m-k} |T(b^m f)(x)| \leq \|b\|_{L^\infty} \sum_{m=0}^k |T(b^m f)(x)| \in L^{p_0},$$

since $b \in L^\infty$, $f \in L_c^\infty$ imply that $b^m f \in L_c^\infty \subset L^{p_0}$ and, by assumption, $T(b^m f) \in L^{p_0}$.

To obtain (ii), we fix $b \in \text{BMO}$ and $f \in L_c^\infty$. Let Q_0 be a cube such that $\text{supp } f \subset Q_0$. We may assume that $b_{Q_0} = 0$ since otherwise we can work with $\tilde{b} = b - b_{Q_0}$ and observe that $T_b^k = T_{\tilde{b}}^k$ and $\|b\|_{\text{BMO}} = \|\tilde{b}\|_{\text{BMO}}$. Note that for all $m = 0, \dots, k$, we have that $|b^m f|$ and $|T(b^m f)|$ are finite almost everywhere since they belong to L^{p_0} .

Let $N > 0$ and define b_N as follows: $b_N(x) = b(x)$ when $-N \leq b(x) \leq N$, $b_N(x) = N$ when $b(x) > N$ and $b_N(x) = -N$ when $b(x) < -N$. Then, it is immediate to see that $|b_N(x) - b_N(y)| \leq |b(x) - b(y)|$ for all x, y . Thus, $\|b_N\|_{\text{BMO}} \leq 2\|b\|_{\text{BMO}}$. As $b_N \in L^\infty$ we can use (6.3) and

$$\|T_{b_N}^k f\|_{L^p(w)} \leq C_0 \|b_N\|_{\text{BMO}}^k \|f\|_{L^p(w)} \leq C_0 2^k \|b\|_{\text{BMO}}^k \|f\|_{L^p(w)} < \infty.$$

To conclude, by Fatou's lemma, it suffices to show that $|T_{b_{N_j}} f(x)| \longrightarrow |T_b^k f(x)|$ for a.e. $x \in \mathbb{R}^n$ and for some subsequence $\{N_j\}_j$ such that $N_j \rightarrow \infty$.

As $|b_N| \leq |b| \in L^p(Q_0)$ for any $1 \leq p < \infty$, the dominated convergence theorem yields that $(b_N)^m f \longrightarrow b^m f$ in L^{p_0} as $N \rightarrow \infty$ for all $m = 0, \dots, k$. Therefore, as T is bounded on L^{p_0} it follows that $T((b_N)^m f - b^m f) \longrightarrow 0$ in L^{p_0} . Thus, there exists a subsequence $N_j \rightarrow \infty$ such that $T((b_{N_j})^m f - b^m f)(x) \longrightarrow 0$ for a.e. $x \in \mathbb{R}^n$ and for all $m = 1, \dots, k$. In this way we obtain

$$\begin{aligned} & |T_{b_{N_j}}^k f(x) - T_b^k f(x)| \lesssim |T((b_{N_j}(x) - b_{N_j})^k - (b(x) - b)^k) f(x)| \\ & \lesssim \sum_{m=0}^k |b_{N_j}(x)|^{k-m} |T((b_{N_j})^m f - b^m f)(x)| + |b_{N_j}(x)^{k-m} - b(x)^{k-m}| |T(b^m f)(x)| \end{aligned}$$

and as desired we get that $|T_{b_{N_j}} f(x)| \longrightarrow |T_b^k f(x)|$ for a.e. $x \in \mathbb{R}^n$. \square

Remark 6.2. The proof just finished can be adapted to the situation of multilinear commutators with no much effort. We just sketch some of the ideas leaving the details to the interested reader. Let us introduce some notation. Given $\vec{b} = (b_1, \dots, b_k)$ we write $\bar{b} = b_1 \cdots b_k$. Let C_j^k , $1 \leq j \leq k$, be the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\} \subset \{1, \dots, k\}$ of j different elements. In this case, we write $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$ and $\bar{b}_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$. We also set $C_0^k = \emptyset$ in which case we understand that $T_{\vec{b}_\sigma} = T$ and $\bar{b}_\sigma = 1$. If $\sigma \in C_j^k$ we set $\sigma' = \{1, \dots, k\} \setminus \sigma$ (note that for $j = 0$ we have $\sigma' = \{1, \dots, k\}$). We need the following multilinear version of (6.1) (see [PT]): given $k \geq 1$, for any $x \in Q$ we have

$$\begin{aligned} \int_Q |f_1 \cdots f_k h|^{p_0} & \leq \|f_1\|_{\exp L}^{p_0} \cdots \|f_k\|_{\exp L, Q}^{p_0} \| |h|^{p_0} \|_{L(\log L)^{k p_0}} \\ & \leq \|f_1\|_{\exp L, Q}^{p_0} \cdots \|f_k\|_{\exp L, Q}^{p_0} M^{[k p_0] + 2}(|h|^{p_0})(x). \end{aligned} \quad (6.4)$$

With this in hand and as done with the regular commutators in Lemma 6.1 the matter can be reduced to the case $b_1, \dots, b_k \in L^\infty$. Once we have that, we combine the ideas from [PT, p. 684] with the proof above. We write $F = |T_{\vec{b}} f(x)|^{p_0} \in L^1$ and observe that $F \leq G_Q + H_Q$ where

$$G_Q = 2^{p_0-1} C \left(\sum_{m=1}^k \sum_{\sigma \in C_m^k} \overline{(b - \lambda)_\sigma} |T_{\vec{b}_\sigma} f(x)| + |T(I - \mathcal{A}_{r(Q)}) f_{Q, \vec{b}}(x)| \right)^{p_0},$$

$H_Q = 2^{p_0-1} |T \mathcal{A}_{r(Q)} f_{Q, \vec{b}}(x)|^{p_0}$, and $f_{Q, \vec{b}} = \prod_{j=1}^k (b_j - (b_j)_{2Q}) f$. Next, one estimates G_Q, H_Q using the same ideas (with (6.4) in place of (6.1)):

$$\int_Q G_Q \lesssim C \left(\sum_{m=1}^k \sum_{\sigma \in C_m^k} M^{[k p_0] + 2}(|T_{\vec{b}_\sigma} f|^{p_0})(x) + M^{[k p_0] + 2}(|f|^{p_0})(x) \right) = G(x),$$

and

$$\left(\int_Q H_Q^q \right)^{\frac{1}{q}} \lesssim M F(x) + \sum_{m=1}^k \sum_{\sigma \in C_m^k} M^{[k p_0] + 2}(|T_{\vec{b}_\sigma} f|^{p_0})(\bar{x}) \equiv C (M F(x) + H_2(\bar{x})).$$

From here the proof proceed as in the case above, noticing that the length of $\vec{b}_{\sigma'}$ is $k - m \leq k - 1$ and so the induction hypothesis applies.

6.3. Proof of Theorem 4.9. Assume that the case $p_0 = 1$ is proved. Then we show that the general case follows automatically. Set $\tilde{p} = p/p_0$, $\tilde{q}_0 = q_0/p_0$ and consider the new family $\tilde{\mathcal{F}}$ consisting of the pairs $(\tilde{f}, \tilde{g}) = (f^{p_0}, g^{q_0})$. Observe that $1 \leq \tilde{p} \leq \tilde{q}_0$ and that $\tilde{p} < \infty$ if $\tilde{q}_0 = \infty$ (that is, $q_0 = \infty$). Besides, (4.1) gives that for all $(\tilde{f}, \tilde{g}) \in \tilde{\mathcal{F}}$

$$\int_{\mathbb{R}^n} \tilde{f}^{\tilde{p}} w \leq C \int_{\mathbb{R}^n} \tilde{g}^{\tilde{p}} w, \quad \text{for all } w \in A_{\tilde{p}} \cap RH_{(\tilde{q}_0/\tilde{p})}'$$

provided the left hand side is finite. Therefore, the same holds for all $1 < \tilde{q} < \tilde{q}_0$ and (4.2) follows with $q = \tilde{q} p_0$.

Assume now that $p_0 = 1$. Observe that the case $q_0 = \infty$ is nothing but Rubio de Francia's extrapolation theorem. So we also impose $q_0 < \infty$. The proof of (4.2) is done on distinguishing the two cases $q < p$ and $q > p$. We use the following notation

$$\phi(q) = \left(\frac{q_0}{q} \right)' (q - 1) + 1.$$

Note that (vii) in Proposition 2.1 says that if $q_0/q > 1$ then $w \in A_q \cap RH_{(\frac{q_0}{q})}'$ if and only if $w^{(\frac{q_0}{q})'} \in A_{\phi(q)}$. We need the following auxiliary result based on Rubio de Francia's algorithm.

Lemma 6.3. *Let $1 < q < q_0$ and w such that $w \in A_q \cap RH_{(\frac{q_0}{q})}'$.*

(a) *If $1 \leq p < q$ and $0 \leq h \in L^{(q/p)'}(w)$, then there exists $H \in L^{(q/p)'}(w)$ such that*

$$(a.1) \quad 0 \leq h \leq H.$$

$$(a.2) \quad \|H\|_{L^{(q/p)'}(w)} \leq 2^{\phi(q)'/(q/p)'} \|h\|_{L^{(q/p)'}(w)}.$$

$$(a.3) \quad H w \in A_p \cap RH_{(\frac{q_0}{p})}' \text{ with constants independent of } h.$$

(b) *If $q < p \leq q_0$ and $0 \leq h \in L^q(w)$, then there exists $H \in L^q(w)$ such that*

$$(b.1) \quad 0 \leq h \leq H.$$

$$(b.2) \quad \|H\|_{L^q(w)} \leq 2^{\phi(q)/q} \|h\|_{L^q(w)}.$$

$$(b.3) \quad H^{-p/(p/q)'} w \in A_p \cap RH_{(\frac{q_0}{p})}' \text{ with constants independent of } h.$$

Admit this result for the moment and continue the proof.

Case $1 \leq p < q$: Let $(f, g) \in \mathcal{F}$ be such that $f, g \in L^q(w)$. Fix w such that $w^{(q_0/q)'} \in A_{\phi(q)}$. Then,

$$\|f\|_{L^q(w)}^p = \|f^p\|_{L^{q/p}(w)} = \sup \int_{\mathbb{R}^n} f^p h w$$

where the supremum is taken over all $0 \leq h \in L^{(q/p)'}(w)$ with $\|h\|_{L^{(q/p)'}(w)} = 1$. Take such a function h and let H be the corresponding function given by (a) in Lemma 6.3. Then by (a.1), (4.1) and (a.3), we have

$$\int_{\mathbb{R}^n} f^p h w \leq \int_{\mathbb{R}^n} f^p H w \leq C \int_{\mathbb{R}^n} g^p H w$$

provided the middle term is finite. This is indeed the case as by Hölder's inequality with $q/p > 1$ and by (a.2)

$$\int_{\mathbb{R}^n} f^p H w \leq \|f\|_{L^q(w)}^p \|H\|_{L^{(q/p)'}(w)} \leq 2^{\phi(q)'/(q/p)'} \|f\|_{L^q(w)}^p < \infty.$$

Note that the same can be done with g and so

$$\int_{\mathbb{R}^n} g^p H w \leq 2^{\phi(q)'/(q/p)'} \|g\|_{L^q(w)}^p.$$

This readily leads to the desired estimate.

Case $q < p \leq q_0$: Let $(f, g) \in \mathcal{F}$ be non-trivial functions such that $f, g \in L^q(w)$. Fix w such that $w \in A_q \cap RH_{(\frac{q_0}{q})}'$. We define

$$h = \frac{f}{\|f\|_{L^q(w)}} + \frac{g}{\|g\|_{L^q(w)}}.$$

Note that $h \in L^q(w)$ and $\|h\|_{L^q(w)} \leq 2$. Let H be the non-negative function given by Lemma 6.3 part (b). Then, using Hölder's inequality with $p/q > 1$ we have

$$\begin{aligned} \|f\|_{L^q(w)} &= \left(\int_{\mathbb{R}^n} f^q H^{-q/(p/q)'} H^{q/(p/q)'} w \right)^{1/q} \\ &\leq \left(\int_{\mathbb{R}^n} f^p H^{-p/(p/q)'} w \right)^{1/p} \left(\int_{\mathbb{R}^n} H^q w \right)^{\frac{1}{q(p/q)'}} \\ &\leq C \left(\int_{\mathbb{R}^n} f^p H^{-p/(p/q)'} w \right)^{1/p}, \end{aligned} \tag{6.5}$$

since (b.2) implies

$$\|H\|_{L^q(w)} \leq 2^{\phi(q)/q} \|h\|_{L^q(w)} \leq 2^{1+\phi(q)/q}.$$

Next, by (b.1) we have $f/\|f\|_{L^q(w)} \leq h \leq H$. Hence, using (b.2) we conclude that

$$\begin{aligned} \left(\int_{\mathbb{R}^n} f^p H^{-p/(p/q)'} w \right)^{1/p} &\leq \|f\|_{L^q(w)} \left(\int_{\mathbb{R}^n} H^{p-p/(p/q)'} w \right)^{1/p} = \|f\|_{L^q(w)} \|H\|_{L^q(w)}^{q/p} \\ &\leq 2^{(q/p)(1+\phi(q)/q)} \|f\|_{L^q(w)} < \infty. \end{aligned}$$

This and (b.3) allow us to employ (4.1). Hence, (6.5) yields

$$\|f\|_{L^q(w)} \leq C \left(\int_{\mathbb{R}^n} g^p H^{-p/(p/q)'} w \right)^{1/p} \leq C \|g\|_{L^q(w)} \|H\|_{L^q(w)}^{q/p} \leq C \|g\|_{L^q(w)},$$

where we have used that g satisfies $g/\|g\|_{L^q(w)} \leq H$ due to (b.1).

To complete the proof it remains to show (4.3). As in [CMP] this follows almost automatically from (4.2) by changing the family \mathcal{F} . Indeed, fix $p_0 < r < q_0$ and given $\{(f_j, g_j)\}_j \subset \mathcal{F}$ we define

$$F_r = \left(\sum_j f_j^r \right)^{1/r}, \quad G_r = \left(\sum_j g_j^r \right)^{1/r}.$$

We consider a new family \mathcal{F}_r consisting of all the pairs (F_r, G_r) . Observe that if $(F_r, G_r) \in \mathcal{F}_r$, using (4.2) with $q = r$, we have

$$\|F_r\|_{L^r(w)}^r = \sum_j \int_{\mathbb{R}^n} f_j^r w \leq C \sum_j \int_{\mathbb{R}^n} g_j^r w = C \|G_r\|_{L^r(w)}^r,$$

for all $w \in A_{r/p_0} \cap RH_{(q_0/r)'}$. This means that the family \mathcal{F}_r satisfies (4.1) with $p = r$. Thus, as we have just obtained, it satisfies (4.2) for all $p_0 < q < q_0$ which turns out to be (4.3). \square

Proof of Lemma 6.3. We first observe that

$$w^{(q_0/q)'} \in A_{\phi(q)} \iff w^{1-q'} = w^{(q_0/q)'(1-\phi(q)')} \in A_{\phi(q)'}$$

Given any weight $0 < u < \infty$ a.e. we define the operator

$$S_u f = \frac{M(fu)}{u}.$$

This operator will be used to perform different versions of Rubio de Francia's algorithm. We start with (a): let $1 \leq p < q$ and $h \in L^{(q/p)'}(w)$. We set $u = w^{q'/\phi(q)'}$. Then, as $w^{1-q'} \in A_{\phi(q)'}$ we have

$$\begin{aligned} \|S_u f\|_{L^{\phi(q)'}(w)}^{\phi(q)'} &= \int_{\mathbb{R}^n} M(fu)^{\phi(q)'} u^{-\phi(q)'} w = \int_{\mathbb{R}^n} M(fu)^{\phi(q)'} w^{1-q'} \\ &\leq C \int_{\mathbb{R}^n} |fu|^{\phi(q)'} w^{1-q'} = C \|f\|_{L^{\phi(q)'}(w)}^{\phi(q)'}. \end{aligned}$$

Let us write $\|S_u\|$ for the norm of S_u as a bounded operator on $L^{\phi(q)'}(w)$. We define the following version of Rubio de Francia's algorithm: for $0 \leq f \in L^{\phi(q)'}(w)$

$$\mathcal{R}f = \sum_{k=0}^{\infty} \frac{S_u^k f}{2^k \|S_u\|^k},$$

where S_u^k is the k -iteration of the operator S_u for $k \geq 1$ and S_u^0 is the identity operator. Given $0 \leq h \in L^{(q/p)'}(w)$ we define

$$H = \mathcal{R}(h^{(q/p)'/\phi(q)'})^{\phi(q)'/(q/p)'}$$

Note that

$$0 \leq f \leq \mathcal{R}f, \quad \|\mathcal{R}f\|_{L^{\phi(q)'}(w)} \leq 2 \|f\|_{L^{\phi(q)'}(w)},$$

and so H satisfies (a.1) and (a.2). Note that we also have

$$S_u(\mathcal{R}f) \leq 2 \|S_u\| \mathcal{R}f \iff M(u \mathcal{R}f) \leq C u \mathcal{R}f \iff u \mathcal{R}f \in A_1$$

and therefore $H^{(q/p)'/\phi(q)'} u \in A_1$ with constant independent of h . Then for all cube $Q \subset \mathbb{R}^n$ (the averages are with respect to Lebesgue measure)

$$\oint_Q H^{(q/p)'/\phi(q)'} u \leq C H^{(q/p)'/\phi(q)'}(x) u(x), \quad \text{a.e. } x \in Q. \quad (6.6)$$

We show (a.3), that is, $(Hw)^{(q_0/p)'} \in A_{\phi(p)}$. If $p = 1$ then (6.6) turns out to be

$$\oint_Q (Hw)^{q'_0} \leq C (H(x)w(x))^{q'_0}, \quad \text{a.e. } x \in Q,$$

that is, $(H w)^{q'_0} \in A_{\phi(1)} = A_1$ as desired. If $p > 1$, using (6.6) we have

$$\begin{aligned} I &= \int_Q (H w)^{(q_0/p)'(1-\phi(p)')} = \int_Q (H w)^{1-p'} \\ &\lesssim \left(\int_Q H^{(q/p)'/\phi(q)'} u \right)^{-\frac{(p'-1)\phi(q)'}{(q/p)'}} \left(\int_Q u^{\frac{(p'-1)\phi(q)'}{(q/p)'}} w^{1-p'} \right) \\ &= \left(\int_Q H^{\frac{q_0(q-1)}{(q_0-1)(q-p)}} u \right)^{-\frac{(q_0-1)(q-p)}{q_0(q-1)(p-1)}} \left(\int_Q w^{1-q'} \right) = I_1 \cdot I_2. \end{aligned}$$

Since $1 < p < q < q_0$ we have that

$$s = \frac{q_0(q-1)}{(q_0-1)(q-p)} \frac{1}{(q_0/p)'} = \frac{(q-1)(q_0-p)}{(q_0-1)(q-p)} > 1, \quad s' = \frac{(q-1)(q_0-p)}{(p-1)(q_0-q)}.$$

Then by Hölder's inequality we obtain

$$\begin{aligned} II &= \int_Q (H w)^{(q_0/p)'} \leq \left(\int_Q H^{(q_0/p)'} s u \right)^{1/s} \left(\int_Q w^{(q_0/p)'} s' u^{1-s'} \right)^{1/s'} \\ &= \left(\int_Q H^{\frac{q_0(q-1)}{(q_0-1)(q-p)}} u \right)^{1/s} \left(\int_Q w^{(q_0/q)'} \right)^{1/s'} = II_1 \cdot II_2. \end{aligned}$$

We gather I_1 and II_1 :

$$I_1^{\phi(p)-1} \cdot II_1 = \left(\int_Q H^{\frac{q_0(q-1)}{(q_0-1)(q-p)}} u \right)^{\frac{1}{s} - (\phi(p)-1) \frac{(q_0-1)(q-p)}{q_0(q-1)(p-1)}} = 1$$

since the outer exponent is equal to 0. On the other hand, for I_2 and II_2 we observe that

$$\begin{aligned} I_2^{\phi(p)-1} \cdot II_2 &= \left(\int_Q w^{1-q'} \right)^{\phi(p)-1} \left(\int_Q w^{(q_0/q)'} \right)^{1/s'} \\ &= \left[\left(\int_Q w^{(q_0/q)'} (1-\phi(q)') \right)^{(\phi(p)-1)s'} \left(\int_Q w^{(q_0/q)'} \right) \right]^{1/s'} \\ &= \left[\left(\int_Q w^{(q_0/q)'} \right) \left(\int_Q w^{(q_0/q)'} (1-\phi(q)') \right)^{\phi(q)-1} \right]^{1/s'} \leq C, \end{aligned}$$

since $w^{(q_0/q)'} \in A_{\phi(q)}$. As a consequence of these latter estimates we can conclude that $(H w)^{(q_0/p)'} \in A_{\phi(p)}$:

$$\begin{aligned} &\left(\int_Q (H w)^{(q_0/p)'} \right) \left(\int_Q (H w)^{(q_0/p)'} (1-\phi(p)') \right)^{\phi(p)-1} = I^{\phi(p)-1} \cdot II \\ &\leq C (I_1^{\phi(p)-1} \cdot II_1) (I_2^{\phi(p)-1} \cdot II_2) \leq C. \end{aligned}$$

We now prove (b). Let $h \in L^q(w)$ and $u = w^{(1-(q_0/q)')/\phi(q)}$. Since $w^{(q_0/q)'} \in A_{\phi(q)}$ we have

$$\begin{aligned} \|S_u f\|_{L^{\phi(q)}(w)}^{\phi(q)} &= \int_{\mathbb{R}^n} M(f u)^{\phi(q)} u^{-\phi(q)} w = \int_{\mathbb{R}^n} M(f u)^{\phi(q)} w^{(q_0/q)'} \\ &\leq C \int_{\mathbb{R}^n} |f u|^{\phi(q)} w^{(q_0/q)'} = C \|f\|_{L^{\phi(q)}(w)}^{\phi(q)}. \end{aligned}$$

Let us write $\|S_u\|$ for the norm of S_u as a bounded operator on $L^{\phi(q)}(w)$. Rubio de Francia's algorithm to be used now is given by

$$\mathcal{R}f = \sum_{k=0}^{\infty} \frac{S_u^k f}{2^k \|S_u\|^k},$$

for $0 \leq f \in L^{\phi(q)}(w)$. Given $0 \leq h \in L^q(w)$ we define

$$H = \mathcal{R}(h^{q/\phi(q)})^{\phi(q)/q}.$$

Note that

$$0 \leq f \leq \mathcal{R}f, \quad \|\mathcal{R}f\|_{L^{\phi(q)}(w)} \leq 2 \|f\|_{L^{\phi(q)}(w)},$$

and so H satisfies (b.1) and (b.2). As in the other case

$$S_u(\mathcal{R}f) \leq 2 \|S_u\| \mathcal{R}f \iff M(u \mathcal{R}f) \leq C u \mathcal{R}f \iff u \mathcal{R}f \in A_1$$

and so $H^{q/\phi(q)} u \in A_1$ with constant independent of h . Thus for all cubes $Q \subset \mathbb{R}^n$

$$\int_Q H^{q/\phi(q)} u \leq C H^{q/\phi(q)}(x) u(x), \quad \text{a.e. } x \in Q. \quad (6.7)$$

We prove (b.3). We do first the case $p = q_0$ and we have to see that $H^{-(q_0-q)} w \in A_{q_0} \cap RH_{\infty}$. Note that (6.7) can be rewritten as

$$\int_Q (H^{q_0-q} w^{-1})^{q'_0-1} \leq C (H^{q_0-q}(x) w^{-1}(x))^{q'_0-1}, \quad \text{a.e. } x \in Q.$$

Then, for almost every $x \in Q$ we have

$$H^{-(q_0-q)}(x) w(x) \lesssim \left(\int_Q (H^{q_0-q} w^{-1})^{q'_0-1} \right)^{-\frac{1}{q'_0-1}} \leq \int_Q H^{-(q_0-q)} w$$

where in the last estimate we have used Jensen's inequality with the convex function $t \mapsto t^{-1/(q'_0-1)}$. This shows that $H^{-(q_0-q)} w \in RH_{\infty}$. On the other hand, we also have

$$\left(\int_Q H^{-(q_0-q)} w \right) \lesssim \left(\int_Q (H^{-(q_0-q)} w)^{1-q'_0} \right)^{-(q_0-1)}$$

which automatically implies that $H^{-(q_0-q)} w \in A_{q_0}$. This completes the case $p = q_0$.

If $p < q_0$, (b.3) is equivalent to $(H^{-p/(p/q)'} w)^{(q_0/p)'} \in A_{\phi(p)}$. By (6.7) we observe that

$$\begin{aligned} I &= \int_Q (H^{-p/(p/q)'} w)^{(q_0/p)'} \lesssim \left(\int_Q H^{q/\phi(q)} u \right)^{-\frac{p(q_0/p)'\phi(q)}{(p/q)'\phi(q)}} \left(\int_Q u^{\frac{p(q_0/p)'\phi(q)}{(p/q)'\phi(q)}} w^{(q_0/p)'} \right) \\ &= \left(\int_Q H^{q/\phi(q)} u \right)^{-\frac{p(q_0/p)'\phi(q)}{(p/q)'\phi(q)}} \left(\int_Q w^{(q_0/q)'} \right) = I_1 \cdot I_2. \end{aligned}$$

Since $1 < q < p < q_0$ we have that

$$s = \frac{q(p-1)}{\phi(q)(p-q)} = \frac{(q_0-q)(p-1)}{(q_0-1)(p-q)} > 1, \quad s' = \frac{(q_0-q)(p-1)}{(q_0-p)(q-1)}.$$

By Hölder's inequality we obtain

$$II = \int_Q (H^{-p/(p/q)'} w)^{(q_0/p)'(1-\phi(p)')} = \int_Q (H^{-p/(p/q)'} w)^{1-p'} = \int_Q H^{\frac{p-q}{p-1}} w^{1-p'}$$

$$\begin{aligned}
&\leq \left(\int_Q H^{\frac{p-q}{p-1}s} u \right)^{1/s} \left(\int_Q w^{(1-p')s'} u^{1-s'} \right)^{1/s'} = \left(\int_Q H^{q/\phi(q)} u \right)^{1/s} \left(\int_Q w^{1-q'} \right)^{1/s'} \\
&= II_1 \cdot II_2.
\end{aligned}$$

For I_1 and II_1 we have

$$I_1 \cdot II_1^{\phi(p)-1} = \left(\int_Q H^{q/\phi(q)} u \right)^{-\frac{p(q_0/p)'\phi(q)}{(p/q)'q} + \frac{\phi(p)-1}{s}} = 1$$

since the outer exponent vanishes. On the other hand, since $w^{(q_0/q)'} \in A_{\phi(q)}$,

$$\begin{aligned}
I_2 \cdot II_2^{\phi(p)-1} &= \left(\int_Q w^{(q_0/q)'} \right) \left(\int_Q w^{1-q'} \right)^{\frac{\phi(p)-1}{s'}} \\
&= \left(\int_Q w^{(q_0/q)'} \right) \left(\int_Q w^{(q_0/q)'(1-\phi(q'))} \right)^{\phi(q)-1} \leq C.
\end{aligned}$$

Collecting the last two estimates we conclude that $(H^{-p/(p/q)'} w)^{(q_0/p)'} \in A_{\phi(p)}$:

$$\begin{aligned}
&\left(\int_Q (H^{-\frac{p}{(p/q)'}} w)^{(q_0/p)'} \right) \left(\int_Q (H^{-\frac{p}{(p/q)'}} w)^{(q_0/p)'(1-\phi(p'))} \right)^{\phi(p)-1} = I \cdot II^{\phi(p)-1} \\
&\leq C (I_1 \cdot II_1^{\phi(p)-1}) (I_2 \cdot II_2^{\phi(p)-1}) \leq C.
\end{aligned}$$

□

Part 2. Calderón-Zygmund methods

7. INTRODUCTION

This section develops a circle of ideas based on the Calderón-Zygmund decomposition. This decomposition was invented in the celebrated article [CZ] to prove that certain singular integrals of convolution type are of weak-type $(1, 1)$. Recall that this decomposition is non-linear and breaks up L^1 functions into good and bad parts. The good part is bounded, while the bad part is a sum of localized and oscillating functions. The oscillation is in the sense of a vanishing mean. This turned out to be a very versatile tool.

The application towards singular integrals was refined in [Hör] with a minimal regularity condition on the kernel matching the oscillation of the bad parts. Then, this was generalized to what is now called Calderón-Zygmund operators, see, e.g, [Mey]. We note that a key ingredient in these arguments is the *a priori* strong or weak-type (p_0, p_0) of the operator for some $p_0 > 1$.

Kernel regularity in some sense is needed for such arguments. After the results obtained in [Heb] and [DR] in a functional calculus setting, a general weak-type $(1, 1)$ criterion is formulated in [DMc]. It still exploits the Calderón-Zygmund decomposition but does not use the oscillation of the bad part. The regularity is expressed in the integrability properties of the kernel of $T(Id - \mathcal{A}_r)$ where \mathcal{A}_r , $r > 0$, is some approximation to the identity. In the classical case, \mathcal{A}_r would be an ordinary mollifying operator with a smooth bump function.

[BK1] develops this idea further for singular “non-integral” operators and establishes a weak-type (p, p) criterion, still assuming of course *a priori* weak-type (p_0, p_0) boundedness for some $p_0 > p$. This result is presented in [Au1] with a simpler and

stronger statement. This is typically an unweighted result but as it works in spaces of homogeneous type, it applies with underlying doubling measure $w(x) dx$, $w \in A_\infty$.

In a sense, we have not much to add to this story. However we present it once again as its argument is needed for further development (Section 8.1). First, a slight strengthening of the hypotheses yields for free boundedness results for commutators of the operator with bounded mean oscillation functions (Section 8.2). Second, we observe that similar unweighted estimates plus an *a priori* weighted weak-type (p_0, p_0) estimate of T implies weighted weak-type (p, p) estimate for a range of p 's with $p < p_0$ depending on the class of weights (Section 8.3).

We also present in Section 9 a result of independent interest but needed in [AM3] concerning a Calderón-Zygmund decomposition for a function in \mathbb{R}^n with gradient controlled in some $L^p(w)$ space for some $p \geq 1$ and doubling weight w supporting a Poincaré inequality. Such a decomposition is used in [Au2] in the Euclidean setting and a similar decomposition appear earlier in [CM] and [BS] for the purpose of real interpolation for Sobolev spaces. See also [AC] for an extension to Riemannian manifolds.

8. EXTENDED CALDERÓN-ZYGMUND THEORY

Except for Section 8.4, we work in \mathbb{R}^n endowed with a Borel doubling measure μ (and we remind the reader that in applications $d\mu(x) = w(x) dx$ with $w \in A_\infty$).

8.1. Blunck and Kunstmann's theorem. We use the following notation: if B is a ball with radius $r(B)$ and $\lambda > 0$, λB denotes the concentric ball with radius $r(\lambda B) = \lambda r(B)$, $C_j(B) = 2^{j+1} B \setminus 2^j B$ when $j \geq 2$, $C_1(B) = 4B$, and

$$\int_{C_j(B)} h d\mu = \frac{1}{\mu(2^{j+1} B)} \int_{C_j(B)} h d\mu. \quad (8.1)$$

We say that the doubling measure μ has doubling order $D > 0$ if $\mu(\lambda B) \leq C_\mu \lambda^D \mu(B)$ for every ball B and every $\lambda > 0$.

The following result appears in a paper by Blunck and Kunstmann [BK1] in a slightly more complicated way with extra hypotheses. This version is due to one of us [Au1].

Theorem 8.1. *Let μ be a doubling Borel measure on \mathbb{R}^n with doubling order D and $1 \leq p_0 < q_0 \leq \infty$. Suppose that T is a sublinear operator of weak-type (q_0, q_0) . Let \mathcal{D} be a subspace of $L^{q_0}(\mu) \cap L^{p_0}(\mu)$ stable under truncation by indicator functions of measurable sets. Let $\{\mathcal{A}_r\}_{r>0}$ be a family of operators acting from \mathcal{D} into $L^{q_0}(\mu)$. Assume that for $j \geq 2$,*

$$\left(\int_{C_j(B)} |T(I - \mathcal{A}_{r(B)})f| d\mu \right) \leq \alpha_j \left(\int_B |f|^{p_0} d\mu \right)^{\frac{1}{p_0}} \quad (8.2)$$

and for $j \geq 1$

$$\left(\int_{C_j(B)} |\mathcal{A}_{r(B)}f|^{q_0} d\mu \right)^{\frac{1}{q_0}} \leq \alpha_j \left(\int_B |f|^{p_0} d\mu \right)^{\frac{1}{p_0}}, \quad (8.3)$$

for all ball B with $r(B)$ its radius and for all $f \in \mathcal{D}$ supported in B . If $\sum_j \alpha_j 2^{Dj} < \infty$ then T is of weak-type (p_0, p_0) and hence T is of strong-type (p, p) for all $p_0 < p < q_0$.

More precisely, there exists a constant C such that for all $f \in \mathcal{D}$,

$$\|Tf\|_{L^p(\mu)} \leq C \|f\|_{L^p(\mu)}.$$

8.2. Commutators with BMO functions: part II. A slightly strengthening of the hypotheses above yields an analog result for the commutators with bounded mean oscillation functions. In this case, since the underlying measure is μ , we work with functions $b \in \text{BMO}(\mu)$ (the definition is as the classical one replacing dx by μ). As μ is a doubling measure, John-Nirenberg's inequality holds in $\text{BMO}(\mu)$. The definition of the commutator is the same as in Section 3.5 but in this case we assume that T is of weak-type (q_0, q_0) in place of being bounded on L^{p_0} . This still guarantees that the commutator is well defined.

Theorem 8.2. *Let μ be a doubling Borel measure on \mathbb{R}^n with doubling order D , $1 \leq p_0 < q_0 \leq \infty$, $b \in \text{BMO}(\mu)$ and $k \in \mathbb{N}$, $k \geq 1$. Suppose that T is a sublinear operator and that T and T_b^m for $m = 1, \dots, k$ are of weak-type (q_0, q_0) . Let $\{\mathcal{A}_r\}_{r>0}$ be a family of operators acting from $L_c^\infty(\mu)$ into $L^{q_0}(\mu)$. Assume that for any ball B with $r(B)$ its radius and for all $f \in L_c^\infty$ supported in B , (8.3) holds, and (8.2) is replaced by the stronger assumption*

$$\left(\int_{C_j(B)} |T(I - \mathcal{A}_{r(B)})f|^r d\mu \right)^{\frac{1}{r}} \leq \alpha_j \left(\int_B |f|^{p_0} d\mu \right)^{\frac{1}{p_0}} \quad (8.4)$$

for some $r > 1$ and all $j \geq 2$. If $\sum_j \alpha_j 2^{Dj} j^k < \infty$ then for all $p_0 < p < q_0$, there exists a constant C (independent of b) such that for all $f \in L_c^\infty(\mu)$,

$$\|T_b^k f\|_{L^p(\mu)} \leq C \|b\|_{\text{BMO}(\mu)}^k \|f\|_{L^p(\mu)}.$$

Remark 8.3. Under the assumptions above, we have $\sum_j \alpha_j 2^{Dj} < \infty$ and consequently, Theorem 8.1 implies that $T = T_b^0$ is of weak-type (p_0, p_0) and hence bounded on $L^p(\mu)$ for all $p_0 < p < q_0$.

Remark 8.4. In applications we will use this result with underlying measure $d\mu(x) = w(x) dx$ with $w \in A_\infty$ and so the weight is hidden in the measure. Let us mention that if $w \in A_\infty$, and so dw is a doubling measure, then the reverse Hölder property yields that $\text{BMO}(w) = \text{BMO}$ with equivalent norms.

Remark 8.5. Our argument requires that the commutators are already weak-type (q_0, q_0) , which could make this result useless. However, this hypothesis can be obtained from Theorem 3.15, see [AM3] for examples of this.

Remark 8.6. As in Remark 3.17, we can also consider multilinear commutators associated with a vector of symbols $\vec{b} = (b_1, \dots, b_k)$ with entries in $\text{BMO}(\mu)$. In this case, we can formulate an analog of Theorem 8.2 proving that $T_{\vec{b}}$ is bounded on $L^p(\mu)$ (see Remark 10.2 below). The precise statement is left to the reader.

8.3. Weighted estimates. We present the following weighted version of Theorem 8.1 which is used in [AM4].

Theorem 8.7. *Let μ be a doubling Borel measure on \mathbb{R}^n , $w \in A_\infty$ with doubling order D_w . Let $\mathcal{D}_1 \subset \mathcal{D}_2$ be subspaces of $L^{q_0}(w)$ and suppose that they are stable under truncation by indicator functions of measurable sets. Let T be a sublinear operator*

defined on \mathcal{D}_2 . Let $\{\mathcal{A}_r\}_{r>0}$ be a family of operators acting from \mathcal{D}_1 into \mathcal{D}_2 . Let $1 \leq p_0 < q_0 \leq \infty$. Assume the following conditions

- (a) There exists $q \in \mathcal{W}_w(p_0, q_0)$ such that T is bounded from $L^q(w)$ to $L^{q,\infty}(w)$.
- (b) For all $j \geq 1$, there exist constants α_j such that for any ball B with $r(B)$ its radius and for any $f \in \mathcal{D}_1$ supported in B ,

$$\left(\int_{C_j(B)} |\mathcal{A}_{r(B)} f|^{q_0} d\mu \right)^{\frac{1}{q_0}} \leq \alpha_j \left(\int_B |f|^{p_0} d\mu \right)^{\frac{1}{p_0}}. \quad (8.5)$$

- (c) There exists $\beta > (s_w)'$, i.e. $w \in RH_{\beta'}$, with the following property: for all $j \geq 2$, there exist constants α_j such that for any ball B with $r(B)$ its radius and for any $f \in \mathcal{D}_1$ supported in B and for $j \geq 2$,

$$\left(\int_{C_j(B)} |T(I - \mathcal{A}_{r(B)})f|^\beta d\mu \right)^{1/\beta} \leq \alpha_j \left(\int_B |f|^{p_0} d\mu \right)^{\frac{1}{p_0}}. \quad (8.6)$$

- (d) $\sum_j \alpha_j 2^{D_w j} < \infty$ for α_j in (b) and (c).

Then T is of strong-type (p, p) with respect to w for all $p \in \mathcal{W}_w(p_0, q_0)$ with $p < q$. More precisely, for such a p , there exists a constant C such that for all $f \in \mathcal{D}_1$,

$$\|Tf\|_{L^p(w)} \leq C \|f\|_{L^p(w)}.$$

Proof. Fix a ball B , f supported in B and let $g = |T(I - \mathcal{A}_{r(B)})f|$ and $h = |\mathcal{A}_{r(B)}f|$. Let $p \in \mathcal{W}_w(p_0, q_0)$ with $p < q$. Since $w \in RH_{(q_0/q)'}$ and $w \in A_{p/p_0}$, (8.5) yields

$$\left(\int_{C_j(B)} h^q dw \right)^{\frac{1}{q}} \lesssim \left(\int_{C_j(B)} h^{q_0} d\mu \right)^{\frac{1}{q_0}} \leq \alpha_j \left(\int_B |f|^{p_0} d\mu \right)^{\frac{1}{p_0}} \lesssim \alpha_j \left(\int_B |f|^p dw \right)^{\frac{1}{p}}.$$

Then as $w \in RH_{\beta'}$ and $w \in A_{p/p_0}$, (8.6) implies

$$\int_{C_j(B)} g dw \lesssim \left(\int_{C_j(B)} g^\beta d\mu \right)^{1/\beta} \leq \alpha_j \left(\int_B |f|^{p_0} d\mu \right)^{\frac{1}{p_0}} \lesssim \alpha_j \left(\int_B |f|^p dw \right)^{\frac{1}{p}}.$$

Thus we are back to the hypothesis of Theorem 8.1 for the doubling measure $w d\mu$ and with exponents $p < q$. This implies that T has weak-type (p, p) with respect to $w d\mu$. As p is arbitrary in an open interval, this implies also strong-type by Marcinkiewicz interpolation theorem. \square

Remark 8.8. Note that (8.5) and (8.6) are unweighted assumptions. Since we assume weighted weak-type (q, q) for T , this seems useless in applications. In fact, it is a good companion of Theorem 3.7. See the application to Riesz transforms on manifolds in [AM4].

Remark 8.9. An examination of the argument shows that if in addition $w \in A_1$ then weighted weak-type holds at $p = p_0$.

Remark 8.10. A simple and special case is the following. If (b), (c) and (d) hold for $p_0 = 1$ and $q_0 = \infty$, then it suffices that (a) holds for some q with $q > r_w$ and the conclusion holds for all $p \in (r_w, q)$.

Remark 8.11. We can obtain a version of Theorem 8.7 for commutators with BMO functions: let $k \geq 1$, $b \in \text{BMO}$ and $w \in A_\infty$. In (a) we further assume that T_b^m , for $m = 1, \dots, k$, are bounded from $L^q(w)$ to $L^{q,\infty}(w)$; the series in (d) becomes

$\sum_j \alpha_j 2^{D_w j} j^k < \infty$; (b), (c) remain the same. In such a case, we show that T_b^k is bounded on $L^p(w)$ for $p < q$, $p \in \mathcal{W}_w(p_0, q_0)$.

The proof is almost identical and we only give the main ideas. The computations for h do not change. To estimate g , in the left-hand side, we need to start with an $L^r(w)$ -norm in place of the $L^1(w)$ -norm. We pick $r > 1$ so that $(s_w)' < \beta/r < \beta$ (note that $(s_w)' < \beta$). This guarantees that $w \in RH_{(\beta/r)'}'$ and from the $L^r(w)$ -norm we pass to the $L^\beta(\mu)$ -norm, after this the desired estimate follows in the same manner. Thus, we can apply Theorem 8.2 to obtain that T_b^k is bounded on $L^{\tilde{p}}(w)$ for all $p < \tilde{p} < q$. As p is arbitrary in an open interval, we conclude that T_b^k is bounded on $L^p(w)$ for all $p < q$ such that $p \in \mathcal{W}_w(p_0, q_0)$.

8.4. Extension to spaces of homogeneous type. The preceding results in this part have been obtained in \mathbb{R}^n equipped with a doubling measure μ . In [AM3] we will use them with μ being either the Lebesgue measure or $d\mu(x) = w(x) dx$ with $w \in A_\infty$ and in [AM4], \mathbb{R}^n will be replaced by a manifold or a Lie group. It is not difficult to see that all the proofs can be adapted to the case of general spaces of homogeneous type (\mathcal{X}, d, μ) (see [CW], [Chr], [Ste]). Precise statements and details are left to the reader.

Let us just make a point about the definition (8.1). It would have looked more natural to use the “true” mean of h over $C_j(B)$ where we divide by $\mu(C_j(B))$ in place of $\mu(2^{j+1}B)$. Our choice is justified partly by the fact that we do not know whether $2B \setminus B$ and $2B$ have comparable mass for all balls, and partly since (fortunately) $\mu(2^{j+1}B)$ is the quantity that appears in computations. Let us note a fairly weak sufficient condition on \mathcal{X} insuring this comparability (which is surely known but we could not find an explicit statement in the literature)

Lemma 8.12. *Assume that there exists $\varepsilon \in (0, 1)$ such that for any ball $B \subset \mathcal{X}$, $(2 - \varepsilon)B \setminus B \neq \emptyset$. Then, $\mu(2B \setminus B) \approx \mu(2B)$ for any ball B , where the implicit constants are independent of B .*

It would be nice to be able to take $\varepsilon = 0$ in the above statement. The argument below shows that $\mu(2B \setminus B) \geq C\mu(2B)$ but with C depending on B . So our statement is the next best thing.

We prove the lemma. It suffices to show that $\mu(2B) \geq \nu\mu(B)$ for some $\nu > 1$. Choose $1 < c < \frac{3}{3-\varepsilon}$. Let B be a ball, x_B its center and r its radius. By hypothesis, there exists $x \in B(x_B, (2 - \varepsilon)cr) \setminus B(x_B, cr)$. Set $B' = B(x, (c - 1)r)$ and note that $B' \subset 2B \setminus B$. Thus $\mu(2B) \geq \mu(B) + \mu(B')$. Now $B \subset \kappa B'$ with $\kappa = \frac{(3-\varepsilon)c}{c-1}$, hence $\mu(B) \leq \mu(\kappa B') \leq C\kappa^D \mu(B')$ where D is the doubling order of μ . Therefore, $\mu(2B) \geq (1 + (C\kappa^D)^{-1})\mu(B)$ as desired.

Remark that if we had assumed that *all* annuli are non-empty then we would obtain for *all* $\lambda > 1$, $\mu(\lambda B) \geq c_\mu \lambda^d \mu(B)$ for some $c_\mu \geq 1$ and $d > 0$ depending on μ . Let us finally observe that Theorems 8.1, 8.2 and 8.7 hold with a -adic annuli for some fixed $a > 1$ instead of dyadic ones. The needed changes in the statements and proofs are left to the reader.

9. ON A SPECIAL CALDERÓN-ZYGMUND DECOMPOSITION

The standard Calderón-Zygmund decomposition of functions allows one to decompose a function into a sum of a good bounded function and bad but localized functions. This decomposition depends on the level sets of the maximal function of f . This is used to prove boundedness results such as Theorem 8.1.

If one wants to prove estimates like $\|Tf\|_p \lesssim \sum_{j=1}^n \|\partial_j f\|_p$ then one observes that the level sets under control are those of the maximal function of each partial $\partial_j f$. But unless one can explicitly express Tf in terms of the functions $\partial_j f$, the decomposition applied to each $\partial_j f$ does not allow to split f as before.

The idea of the following lemma, which is applied in [AM3], is to split f according to some information on its gradient. This was done in [Au1] for Lebesgue measure in \mathbb{R}^n . We extend it to a class of doubling measures.

Proposition 9.1. *Let $n \geq 1$ and $1 \leq p < \infty$. Let $w \in L^1_{\text{loc}}(\mathbb{R}^n)$, $w > 0$ a.e., be such that $d\mu = w dx$ is a Borel doubling measure (here we do not need that w is a Muckenhoupt weight). Assume that the measure μ supports an L^p Poincaré inequality, that is,*

$$\left(\int_B |f - m_B f|^p d\mu \right)^{\frac{1}{p}} \leq C r(B) \left(\int_B |\nabla f|^p d\mu \right)^{\frac{1}{p}} \quad (9.1)$$

for all locally Lipschitz functions f and all balls B with radius $r(B)$. Here $m_B f$ is the average of f with respect to μ on B . Assume that $f \in \mathcal{S}$ is such that $\|\nabla f\|_{L^p(\mu)} < \infty$.[†] Let $\alpha > 0$. Then, one can find a collection of balls $\{B_i\}_i$, smooth functions $\{b_i\}_i$ and a function $g \in L^1_{\text{loc}}(\mathbb{R}^n, \mu)$ such that

$$f = g + \sum_i b_i \quad (9.2)$$

and the following properties hold:

$$|\nabla g(x)| \leq C\alpha, \quad \text{for } \mu\text{-a.e. } x \quad (9.3)$$

$$\text{supp } b_i \subset B_i \quad \text{and} \quad \int_{B_i} |\nabla b_i|^p d\mu \leq C\alpha^p \mu(B_i), \quad (9.4)$$

$$\sum_i \mu(B_i) \leq C\alpha^{-p} \int_{\mathbb{R}^n} |\nabla f|^p d\mu, \quad (9.5)$$

$$\sum_i \chi_{B_i} \leq N, \quad (9.6)$$

where C and N depends only on dimension, the doubling constant of μ and p . Assuming furthermore that μ supports an $L^p - L^q$ Poincaré inequality with $p \leq q < \infty$, that is,

$$\left(\int_B |f - m_B f|^q d\mu \right)^{\frac{1}{q}} \leq C r(B) \left(\int_B |\nabla f|^p d\mu \right)^{\frac{1}{p}} \quad (9.7)$$

[†]We avoid here regularity issues by taking a smooth f .

[‡]The gradient of g exists μ -almost everywhere, that is almost everywhere for the Lebesgue measure. In fact, a similar argument shows that g is almost everywhere equal to a Lipschitz function \tilde{g} . Hence, ∇g coincide almost everywhere with the distributional gradient of \tilde{g} .

for all f locally Lipschitz and all ball B . Then

$$\left(\int_{B_i} |b_i|^q d\mu \right)^{\frac{1}{q}} \lesssim \alpha r(B_i). \quad (9.8)$$

Since A_p weights support an $L^p - L^q$ Poincaré inequality for some $q > p$, the latter result applies to any $w \in A_\infty$ and $p > r_w$.

10. PROOFS OF THE MAIN RESULTS

We prove Theorem 8.1, Theorem 8.2, and Proposition 9.1.

10.1. Proof of Theorem 8.1. We follow closely the proof in [Au1] (we include it since it will be needed for the next section). By Marcinkiewicz interpolation theorem, it suffices to show that T is of weak-type (p_0, p_0) . Let $f \in \mathcal{D}$ (so $f \in L^{p_0}(\mu)$) and $\alpha > 0$. By the Calderón-Zygmund decomposition (see [CW] or [Ste]) for $|f|^{p_0}$ at height α^{p_0} it follows that there exist a collection of balls $\{B_i\}_i$ and functions $g, \{h_i\}_i$ such that $f = g + \sum_i h_i$ and the following properties hold:

$$\|g\|_{L^\infty(\mu)} \leq C \alpha, \quad (10.1)$$

$$\text{supp } h_i \subset B_i, \quad \left(\int_{B_i} |h_i|^{p_0} d\mu \right)^{\frac{1}{p_0}} \leq C \alpha, \quad (10.2)$$

$$\sum_i \mu(B_i) \leq C \alpha^{-p_0} \int_{\mathbb{R}^n} |f|^{p_0} d\mu, \quad (10.3)$$

$$\sum_i \chi_{B_i} \leq N, \quad (10.4)$$

where C and N depends on μ, n and p_0 . We write $r_i = r(B_i)$ and control Tf by

$$|Tf| \leq |Tg| + \left| T \left(\sum_i \mathcal{A}_{r_i} h_i \right) \right| + \sum_i |T(I - \mathcal{A}_{r_i})h_i| = F_1 + F_2 + F_3.$$

We estimate $\mu\{F_i > \alpha/3\}$. For F_1 , since T is of weak-type (q_0, q_0) and (10.1)

$$\mu\{F_1 > \alpha/3\} \lesssim \frac{1}{\alpha^{q_0}} \int_{\mathbb{R}^n} |g|^{q_0} d\mu \lesssim \frac{1}{\alpha^{p_0}} \int_{\mathbb{R}^n} |g|^{p_0} d\mu \lesssim \frac{1}{\alpha^{p_0}} \int_{\mathbb{R}^n} |f|^{p_0} d\mu, \quad (10.5)$$

where we have used that (10.4), (10.2), (10.3) yield

$$\int_{\mathbb{R}^n} \left| \sum_i h_i \right|^{p_0} d\mu \lesssim \sum_i \int_{B_i} |h_i|^{p_0} d\mu \lesssim \alpha^{p_0} \sum_i \mu(B_i) \lesssim \int_{\mathbb{R}^n} |f|^{p_0} d\mu.$$

For F_2 , we first use that T is of weak-type (q_0, q_0) ,

$$\mu\{F_2 > \alpha/3\} \lesssim \frac{1}{\alpha^{q_0}} \int_{\mathbb{R}^n} \left| \sum_i \mathcal{A}_{r_i} h_i \right|^{q_0} d\mu. \quad (10.6)$$

To compute the L^{q_0} -norm we dualize against $0 \leq u \in L^{q'_0}(\mu)$ with $\|u\|_{L^{q'_0}(\mu)} = 1$. We use (8.3), (10.2), (10.4)

$$\int_{\mathbb{R}^n} \left| \sum_i \mathcal{A}_{r_i} h_i \right| u d\mu \lesssim \sum_i \sum_{j=1}^{\infty} 2^{jD} \mu(B_i) \left(\int_{C_j(B_i)} |\mathcal{A}_{r_i} h_i|^{q_0} d\mu \right)^{\frac{1}{q_0}} \left(\int_{2^{j+1} B_i} u^{q'_0} d\mu \right)^{\frac{1}{q'_0}}$$

$$\begin{aligned}
&\lesssim \sum_i \sum_{j=1}^{\infty} 2^{jD} \mu(B_i) \alpha_j \left(\int_{B_i} |h_i|^{p_0} d\mu \right)^{\frac{1}{p_0}} \operatorname{ess\,inf}_{y \in B_i} M_{\mu}(u^{q'_0})^{\frac{1}{q'_0}}(y) \\
&\lesssim \alpha \int_{\mathbb{R}^n} \sum_i \chi_{B_i} M_{\mu}(u^{q'_0})^{\frac{1}{q'_0}} d\mu \lesssim \alpha \int_{\cup_i B_i} M_{\mu}(u^{q'_0})^{\frac{1}{q'_0}} d\mu \\
&\lesssim \alpha \mu(\cup_i B_i)^{\frac{1}{q_0}} \|u^{q'_0}\|_{L^1(\mu)}^{\frac{1}{q'_0}} = \alpha \mu(\cup_i B_i)^{\frac{1}{q_0}}, \tag{10.7}
\end{aligned}$$

where we have used Kolmogorov's lemma and the weak-type $(1, 1)$ for the Hardy-Littlewood maximal function M_{μ} (this idea is borrowed from [HM]). Next, we take the supremum on u and plug the obtained estimate into (10.6):

$$\mu\{F_2 > \alpha/3\} \lesssim \mu(\cup_i B_i) \lesssim \frac{1}{\alpha^{p_0}} \int_{\mathbb{R}^n} |f|^{p_0} d\mu, \tag{10.8}$$

where we have used (10.3). Next, we consider F_3 . By (8.2), (10.2) and (10.3)

$$\begin{aligned}
\mu((\mathbb{R}^n \setminus \cup_i 4B_i) \cap \{F_3 > \alpha/3\}) &\leq \frac{3}{\alpha} \sum_i \int_{\mathbb{R}^n \setminus 4B_i} |T(I - \mathcal{A}_{r_i})h_i| d\mu \\
&\lesssim \frac{1}{\alpha} \sum_i \sum_{j=2}^{\infty} 2^{jD} \mu(B_i) \left(\int_{C_j(B_i)} |T(I - \mathcal{A}_{r_i})h_i| d\mu \right) \\
&\lesssim \frac{1}{\alpha} \sum_i \sum_{j=2}^{\infty} 2^{jD} \mu(B_i) \alpha_j \left(\int_{B_i} |h_i|^{p_0} d\mu \right)^{\frac{1}{p_0}} \lesssim \frac{1}{\alpha^{p_0}} \int_{\mathbb{R}^n} |f|^{p_0} d\mu. \tag{10.9}
\end{aligned}$$

Gathering (10.5), (10.8), (10.9), and using (10.3) we conclude that

$$\mu\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\} \lesssim \frac{1}{\alpha^{p_0}} \int_{\mathbb{R}^n} |f|^{p_0} d\mu.$$

10.2. Proof of Theorem 8.2. The basic ingredient is the following consequence of John-Nirenberg's inequality: for any ball B , $0 < s < \infty$ and $j \geq 0$,

$$\left(\int_{2^j B} |b - b_B|^s d\mu \right)^{\frac{1}{s}} \lesssim (1+j) \|b\|_{\text{BMO}(\mu)}. \tag{10.10}$$

Lemma 10.1. *Assume (8.3) and (8.4) of Theorem 8.2. Let $p_0 < p < q < q_0$. Let $b \in L^{\infty}(\mu)$ with $\|b\|_{\text{BMO}(\mu)} = 1$. Then for all ball B with radius r , all functions f supported in B and $m \in \mathbb{N}$, $m \geq 1$,*

$$\left(\int_B |(b - b_{4B})^m f|^{p_0} d\mu \right)^{\frac{1}{p_0}} \lesssim \left(\int_B |f|^p d\mu \right)^{\frac{1}{p}}, \tag{10.11}$$

for $j \geq 1$,

$$\left(\int_{C_j(B)} |(b - b_{4B})^m \mathcal{A}_r f|^q d\mu \right)^{\frac{1}{q}} \lesssim j^m \alpha_j \left(\int_B |f|^{p_0} d\mu \right)^{\frac{1}{p_0}} \tag{10.12}$$

and for $j \geq 2$,

$$\int_{C_j(B)} |(b - b_{4B})^m T(I - \mathcal{A}_r)f| d\mu \lesssim j^m \alpha_j \left(\int_B |f|^{p_0} d\mu \right)^{\frac{1}{p_0}}, \tag{10.13}$$

where the constants involved are independent of b and f .

The proof of (10.11) is a direct application of Hölder inequality and (10.10). Next, using that $q < q_0$, (10.12) follows from Hölder inequality, (8.3) and (10.10). Eventually, (10.13) is a consequence of Hölder inequality, (8.4) as $r > 1$ and (10.10).

We begin the proof of Theorem 8.2. As before it is enough to consider the case $b \in L^\infty(\mu)$ obtaining the desired estimates with a constant independent of b . Let us observe that here we assume that T is of weak-type (q_0, q_0) in place of being bounded on L^{q_0} . This changes slightly Lemma 6.1. Namely, in (a) one obtains that $T_b^k f \in L^{q_0, \infty}(\mu)$. The proof of (b) changes in the following way: one shows that $T((b_N)^m f - b^m f) \rightarrow 0$ in $L^{q_0, \infty}(\mu)$ which also implies the convergence almost everywhere for a subsequence. From here the proof can be carried out in the same manner.

When $b \in L^\infty(\mu)$, all the formal computations below make sense. Notice that by homogeneity, it suffices to consider the case $\|b\|_{\text{BMO}(\mu)} = 1$. By Marcinkiewicz interpolation theorem, it suffices to show that T_b^k is of weak-type (p, p) for all $p_0 < p < q_0$ because T_b^k is sublinear. We proceed by induction and assume that we have proved that T_b^m is of weak-type (p, p) for all $p_0 < p < q_0$ and $m = 0, \dots, k-1$, the case $m = 0$ being covered by Theorem 8.1.

Fix p so that $p_0 < p < q_0$ and let q with $p < q < q_0$. Let $f \in L_c^\infty$ (so $f \in L^p(\mu)$) and $\alpha > 0$. By the Calderón-Zygmund decomposition (see [CW] or [Ste]) for $|f|^p$ at height α^p it follows that there exist a collection of balls $\{B_i\}_i$, a collection of functions $\{h_i\}_i$ and a function g such that $f = g + \sum_i h_i$ and (10.1), (10.2), (10.3) (10.4) hold with p in place of p_0 . We wish to estimate $\mu\{|T_b^k f| > \alpha\}$. First, we have

$$|T_b^k f| \leq |T_b^k g| + \left| T_b^k \left(\sum_i h_i \right) \right|.$$

By the weak-type (q_0, q_0) of T_b^k ,

$$\mu\{|T_b^k g| > \alpha/2\} \lesssim \frac{1}{\alpha^{q_0}} \int_{\mathbb{R}^n} |g|^{q_0} d\mu \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |f|^p d\mu, \quad (10.14)$$

where the last inequality follows as in (10.5). Next, set $h_{i,b}^m = (b_{4B_i} - b)^m h_i$ and $r_i = r(B_i)$. Then

$$\begin{aligned} \left| T_b^k \left(\sum_i h_i \right) (x) \right| &\leq \sum_{m=0}^k C_{k,m} \left| T \left(\sum_i (b(x) - b_{4B_i})^{k-m} \mathcal{A}_{r_i} h_{i,b}^m \right) (x) \right| \\ &\quad + \sum_{m=0}^k C_{k,m} \sum_i |b(x) - b_{4B_i}|^{k-m} \left| T \left((I - \mathcal{A}_{r_i}) h_{i,b}^m \right) (x) \right| \end{aligned}$$

The m -th term in the first sum is bounded by $\sum_{\ell=0}^{k-m} c_\ell^m F_{m,\ell}(x)$ with

$$F_{m,\ell}(x) = \left| T_b^{k-m-\ell} \left(\sum_i (b - b_{4B_i})^\ell \mathcal{A}_{r_i} h_{i,b}^m \right) (x) \right|.$$

Fix $\ell = m = 0$ and A some large number depending just on k . Then the estimate of $\mu\{F_{0,0} > \alpha/A\}$ is done as for the term F_2 in the proof of Theorem 8.1, using the weak-type (q_0, q_0) of T_b^k . Next, fix ℓ, m with $m+\ell > 0$. Then, the induction hypothesis implies that $T_b^{k-m-\ell}$ is of weak-type (q, q) . Hence, the estimate of $\mu\{F_{m,\ell} > \alpha/A\}$ is done as for the term F_2 in the proof of Theorem 8.1, by replacing q_0 by q and using (10.12) with $f = h_{i,b}^m$ and then (10.11) with $f = h_i$.

It remains to estimate $\mu\{G_{m,\ell} > \alpha/A\}$ with

$$G_{m,\ell}(x) = \sum_i |b(x) - b_{4B_i}|^{k-m} \left| T \left((I - \mathcal{A}_{r_i}) h_{i,b}^m \right) (x) \right|.$$

We proceed as for the term F_3 in the proof of Theorem 8.1, using (10.13) with $f = h_{i,b}^m$ and then (10.11) with $f = h_i$. We leave details to the reader.

Remark 10.2. The latter argument can be carried out for the multilinear commutators introduced above. We give some of the ideas leaving the precise computations to the reader. As before, it suffices to consider the case $b_m \in L^\infty$ with $\|b_m\|_{\text{BMO}(\mu)} = 1$ for all $1 \leq m \leq k$. Given $\sigma \subset \{1, \dots, k\}$, we write $\pi_{i,\vec{b}_\sigma} = \prod_{j \in \sigma} (b_j - (b_j)_{4B_i})$ and $h_{i,\vec{b}_\sigma} = h_i \pi_{i,\vec{b}_\sigma}$. Here, when $\sigma = \emptyset$ we understand that $\pi_{i,\vec{b}_\sigma} = 1$ and $h_{i,\vec{b}_\sigma} = h_i$. Thus, combining the preceding ideas with [PT, p. 684] we have

$$|T_{\vec{b}} f| \leq |T_{\vec{b}} g| + \sum_{\sigma_1, \sigma_2, \sigma_3} \left| T_{\vec{b}_{\sigma_1}} \left(\sum_i \pi_{i,\vec{b}_{\sigma_2}} \mathcal{A}_{r_i} h_{i,\vec{b}_{\sigma_3}} \right) \right| + \sum_{\sigma_1, \sigma_2} \sum_i |\pi_{i,\vec{b}_{\sigma_1}}| |T(I - \mathcal{A}_{r_i}) h_{i,\sigma_2}|,$$

where the first sum (resp. the second sum) runs over all partitions of $\{1, \dots, k\}$ in three (resp. two) pairwise disjoint sets $\sigma_1, \sigma_2, \sigma_3$ (resp. σ_1, σ_2).

The estimate for the first term is obtained as in (10.14). The second term is treated as $F_{m,l}$ above (notice that the case $\sigma_1 = \{1, \dots, k\}$, $\sigma_2 = \sigma_3 = \emptyset$ is handled differently as happened before). Finally, the third term is estimated as $G_{m,l}$ above. Full details are left to the reader.

10.3. Proof of Proposition 9.1. Let $\Omega = \{x \in \mathbb{R}^n : M_\mu(|\nabla f|^p)(x) > \alpha^p\}$ where M_μ is the uncentered maximal operator over cubes[†] of \mathbb{R}^n with respect to μ . If Ω is empty, then set $g = f$. Otherwise, since μ is doubling it follows that M_μ is of weak-type (p, p) and so

$$|\Omega| \leq C\alpha^{-p} \int_{\mathbb{R}^n} |\nabla f|^p d\mu.$$

Let F be the complement of Ω . By the Lebesgue differentiation theorem, $|\nabla f| \leq \alpha$ μ -almost everywhere on F .

Lemma 10.3. *One can redefine f on a μ -null set of F so that for all $x \in F$, and for all cube Q centered at x ,*

$$|f(x) - m_Q f| \leq C\alpha\ell(Q) \quad (10.15)$$

where $\ell(Q)$ is the sidelength of Q . Furthermore, for all $x, y \in F$,

$$|f(x) - f(y)| \leq C\alpha|x - y|. \quad (10.16)$$

The constant C depends only on dimension, the doubling constant of μ and p .

Proof of Lemma 10.3. Let x be a point in F . Fix a cube Q with center x and let Q_k be co-centered cubes with $\ell(Q_k) = 2^{-k} \ell(Q)$ for $k \geq 1$. Then, by Poincaré's inequality

$$|m_{Q_{k+1}} f - m_{Q_k} f| \lesssim \int_{Q_k} |f - m_{Q_k} f| d\mu \lesssim \ell(Q_k) \left(\int_{Q_k} |\nabla f|^p d\mu \right)^{\frac{1}{p}} \lesssim 2^{-k} \ell(Q) \alpha \quad (10.17)$$

since $x \in Q_k \cap F$. This easily implies that $\{m_{Q_k} f\}_{k \geq 1}$ is a Cauchy sequence and so it converges as $k \rightarrow \infty$ or what is the same as $\ell(Q_k) \rightarrow 0$. The Lebesgue differentiation

[†]We freely change balls to cubes.

theorem implies that $m_{Q_k}f \rightarrow f(x)$ whenever x is a Lebesgue point of f , that is μ -almost everywhere. If x is not a Lebesgue point, it is easy to show that $\lim m_{Q_k}f$ does not depend on Q (the original cube). Hence, we redefine $f(x)$ as the value of this limit. With this new definition, summing over $k \geq 1$ on (10.17) one gets (10.15).

To see (10.16), let $x, y \in F$ and Q_x be the cube centered at x with sidelength $2|x - y|$ and Q_y be the cube centered at y with sidelength $4|x - y|$. It is easy to see that $Q_x \subset Q_y$. As in (10.17), one can see that $|m_{Q_x}f - m_{Q_y}f| \leq C\alpha|x - y|$. Hence by the triangle inequality and (10.15), one obtains (10.16) readily. \square

Let us continue the proof of Lemma 9.1. Let $\{Q_i\}_i$ be a Whitney decomposition of Ω by dyadic cubes. Hence, Ω is the disjoint union of the Q_i 's, the cubes $2Q_i \subset \Omega$ have bounded overlap, and the cubes $4Q_i$ intersect F . As usual, λQ is the cube co-centered with Q with sidelength $\ell(\lambda Q) = \lambda \ell(Q)$. Hence (9.5) and (9.6) are satisfied by the cubes $2Q_i$.

Let us now define the functions b_i and show (9.4). Let $\{\mathcal{X}_i\}_i$ be a partition of unity on Ω associated to the covering $\{Q_i\}_i$ so that for each i , \mathcal{X}_i is a C^∞ function supported in $2Q_i$ with $\|\mathcal{X}_i\|_\infty + \ell_i \|\nabla \mathcal{X}_i\|_\infty \leq c(n)$, ℓ_i being the sidelength of Q_i . Set

$$b_i = (f - m_{2Q_i}f) \mathcal{X}_i.$$

It is clear that b_i is supported in $2Q_i$. Since $\nabla((f - m_{2Q_i}f)\mathcal{X}_i) = \mathcal{X}_i \nabla f + (f - m_{2Q_i}f) \nabla \mathcal{X}_i$, we have by the L^p Poincaré inequality, the fact that the average of $|\nabla f|^p$ on $4Q_i$ is controlled by α^p (since $4Q_i$ meets F) and the doubling property that

$$\int_{2Q_i} |\nabla((f - m_{2Q_i}f)\mathcal{X}_i)|^p d\mu \leq C\alpha^p \mu(2Q_i).$$

Thus (9.4) is proved.

It remains to obtain (9.2) and (9.3). To do so, we introduce an auxiliary function $h = \sum_i m_{2Q_i}f \nabla \mathcal{X}_i$, for which we claim that $h \leq C\alpha$ on \mathbb{R}^n . First, note that this sum is locally finite in Ω and vanishes on F , hence h well-defined on \mathbb{R}^n . Note also that $\sum_i \mathcal{X}_i$ is 1 on Ω and 0 on F . Since it is also locally finite we have $\sum_i \nabla \mathcal{X}_i = 0$ in Ω . Fix $x \in \Omega$. Let Q_j be the Whitney cube containing x and let I_x be the set of indices i such that $x \in 2Q_i$. We know that $\#I_x \leq N$. Also for $i \in I_x$ we have that $C^{-1}\ell_i \leq \ell_j \leq C\ell_i$ where the constant C depends only on dimension (see [Ste]). We also have $|m_{2Q_i}f - m_{2Q_j}f| \leq C\ell_j\alpha$ (embed $2Q_i$ and $2Q_j$ in some dilate of Q_j and apply Poincaré's inequality as in (10.17) and the definition of F). Hence,

$$|h(x)| = \left| \sum_{i \in I_x} (m_{2Q_i}f - m_{2Q_j}f) \nabla \mathcal{X}_i(x) \right| \leq C \sum_{i \in I_x} |m_{2Q_i}f - m_{2Q_j}f| \ell_i^{-1} \leq CN\alpha.$$

We are ready to prove (9.2) and (9.3). Set $g = f - \sum b_i$. This function is defined μ -almost everywhere, hence (9.2) trivially holds. Next, we claim that $\nabla g = \mathbf{1}_F(\nabla f) + h$ μ -almost everywhere where $\mathbf{1}_E$ is the indicator function of a set E . Admitting this, for μ -a.e. $x \in F$, we have that $|\nabla g(x)| = |\nabla f(x)| \leq M_\mu(|\nabla f|^p)(x)^p \leq \alpha$, and for μ -a.e. $x \in \Omega$, $|\nabla g(x)| = |h(x)| \leq CN\alpha$. To conclude the proof of (9.3), it remains to see the claim. First, observe that $\sum_i b_i$ converges in $L^p_{\text{loc}}(\mathbb{R}^n, \mu)$. Indeed, fix a compact set K and observe that the sidelengths of the cubes Q_i meeting K are bounded. Since $\|b_i\|_{L^p(\mu)}^p \leq C\ell_i^p \alpha^p \mu(Q_i)$ and $\sum_i \mu(Q_i) < \infty$, we obtain convergence of the series in $L^p(K, \mu)$ from the bounded overlap property of the Q_i 's. Next, it follows from (9.4),

(9.5) and (9.6) that $\sum_i |\nabla b_i|$ converges in $L^p(\mathbb{R}^n, \mu)$. We invoke [FHK, Corollary 11] (this is where we use that μ is given by a weight) which implies that ∇g exists almost everywhere (which is the same as μ -almost everywhere by the assumption on the weight) and is given by $\nabla f - \sum_i \nabla b_i$. But as $\sum_i \nabla \mathcal{X}_i(x) = 0$ for $x \in \Omega$, we have

$$\nabla f = \mathbf{1}_F(\nabla f) + \mathbf{1}_\Omega(\nabla f) = \mathbf{1}_F(\nabla f) + h + \sum_i \nabla b_i \quad \mu\text{-a.e.},$$

and the claim follows.

It remains to prove (9.8) assuming an $L^p - L^q$ Poincaré inequality. By the definition of b_i and similar computation as above,

$$\left(\int_{Q_i} |b_i|^q d\mu \right)^{\frac{1}{q}} \lesssim \ell_i \left(\int_{2Q_i} |\nabla f|^p d\mu \right)^{\frac{1}{p}} \lesssim \ell_i \alpha.$$

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